CK-12 Trigonometry

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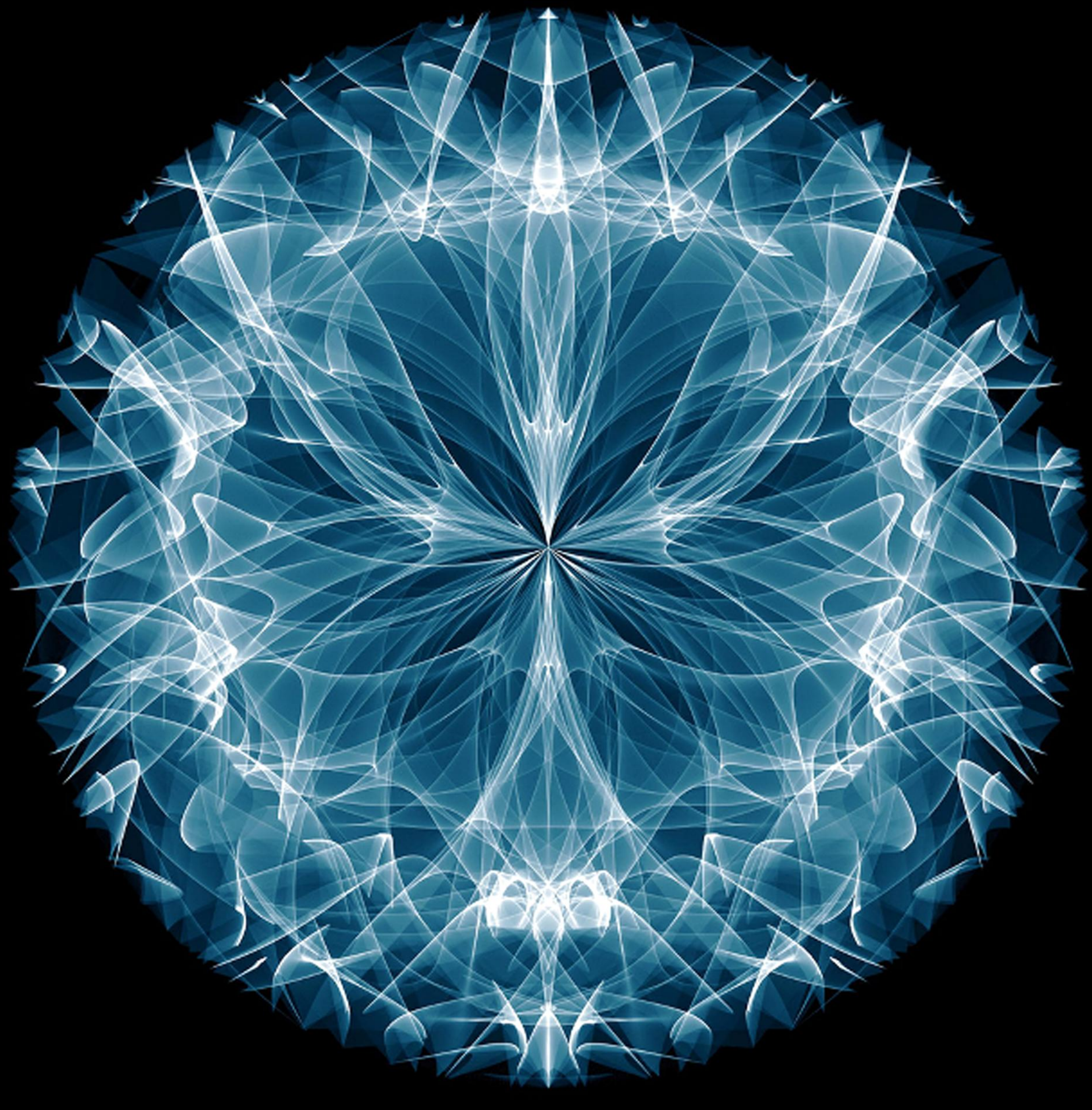

Trigonometry Teacher's Edition - Solution Key

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CHAPTER 1 Trigonometry and Right Angles - Solution Key

CHAPTER OUTLINE

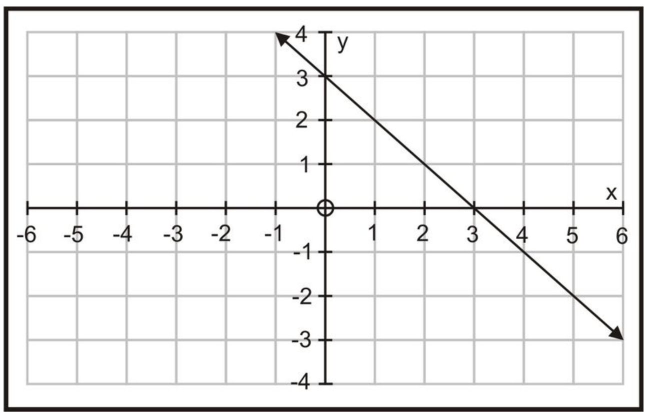
1.1 Trigonometry and Right Angles

Basic Functions

Review Exercises:

1. a) This relation is not a function. The $x-$ value of 1 is paired with two $y-$ values: 5 and 7 .

b) This relation is a function. Any vertical line will cross the graph of $y = 3 - x$ only once. Each $x-$ value is paired with one and only one $y-$ value.



x	-3	-2	-1	0	1	2	3
y	6	5	4	3	2	1	0

c) This relation is not a function. Any vertical line will cross the graph more than once.

2. a)

$$\text{distance} = \text{rate} \cdot \text{time}$$
$$d = 95t$$

b) This situation is direct variation because as the time increases the distance increases at the same rate.

c)

$$d = 95t \rightarrow \text{general equation}$$
$$d = 95 \text{ miles/hr } (3\text{hr}) \rightarrow \text{given } t = 3 \text{ hours}$$
$$d = 285 \text{ miles.}$$

3. a) $y = mx + b$. y represents the cost of beginning the business; m represents the cost of each wooden frame (x) and b represents the initial output of money $(0, 100)$.

$$y = 2x + 100$$
$$c(x) = 2x + 100 \rightarrow y = 2x + 100 \text{ written as a function.}$$

b) $y = mx + b$. y represents the revenue; m represents the selling price of each picture frame and b represents any other revenue which in this case is zero.

$$y = 10x$$
$$R(x) = 10x \rightarrow y = 10x \text{ written as a function.}$$

c)

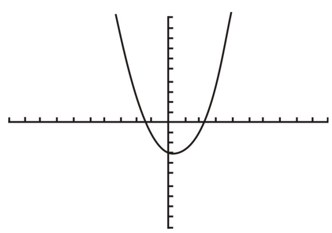
$$P(x) = R(x) - C(x) \rightarrow \text{The profit } P(x) \text{ is the difference between the revenue} R(x) \text{ and the cost } C(x)$$
$$P(x) = 10x - (2x + 100)$$
$$P(x) = 10x - 2x - 100$$
$$P(x) = 8x - 100$$

4. a) The function defined by the equation $f(x) = x^2 - x - 3$ is of the general form of a quadratic function.

b) The domain of the function is: $\{xIx\varepsilon R\}$

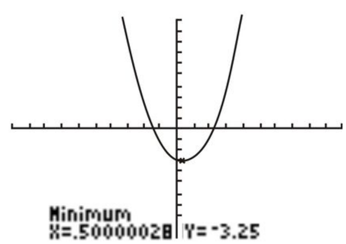
The range of the function is: $\{yIy \ge -3.25, y\varepsilon R\}$

c) Using the TI – 83 to graph $f(x) = x^2 - x - 3$

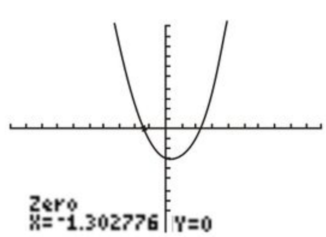


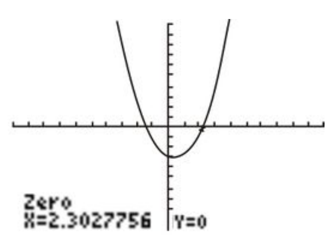
The coordinates of the vertex and the $x-$ intercepts can be determined by using the 2^{nd} Trace function:

Vertex



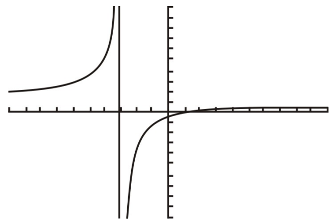
$x-$ **intercepts**





The vertex is $(5.0, -3.25)$ and the $x-$ intercepts are $(-1.3, 0)$ and $(2.3, 0)$.

5. a) Using the TI – 83 to graph $y = \frac{x-2}{x+3}$:



The asymptotes are $x = -3$ and $y = 1$

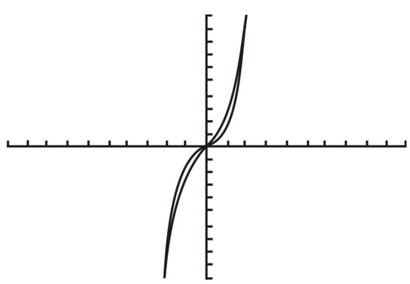
6. a) $c = \frac{1}{p}(500) \rightarrow$ the cost per person (c) of renting a party room varies inversely with the number of people who attend and the initial cost of renting the room (500)

b)

$$c = \frac{1}{p}(500) \rightarrow \text{ given } p = 32$$
$$c = \frac{1}{32}(500)$$
$$c = \$15.63$$

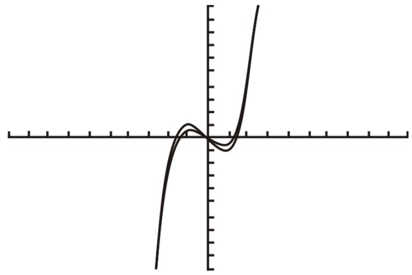
7. a) Using the TI – 83 to graph:

$$y = x^3$$
$$y = x^3 + x$$
$$y = x^3 + 2x$$



The equations with positive coefficients look more and more like $y = x^3$, as the coefficient gets larger.

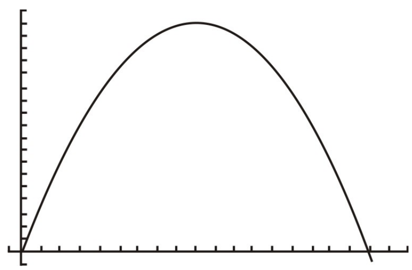
$$y = x^3$$
$$y = x^3 - x$$
$$y = x^3 - 2x$$



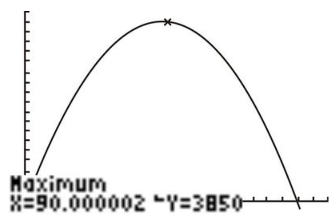
The equations with negative coefficients have local maximums and minimums.

Decreasing the coefficient increases the size of the" hill" and the "valley."

8. a) Using the TI – 83 to graph the function $p(x) = -.5x^2 + 90x - 200$:

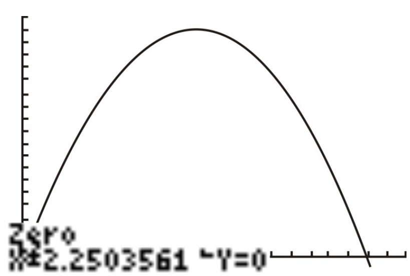


The number of units that must be sold to attain the maximum profit is the vertex of the parabola. Use 2[nd] Trace

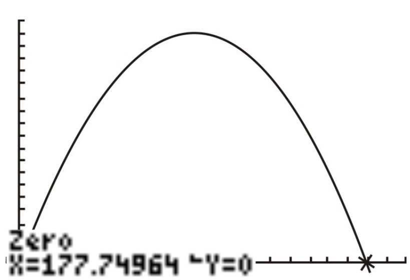


The maximum profit is $3850 with 90 units being sold.

The $x-$ intercepts are:

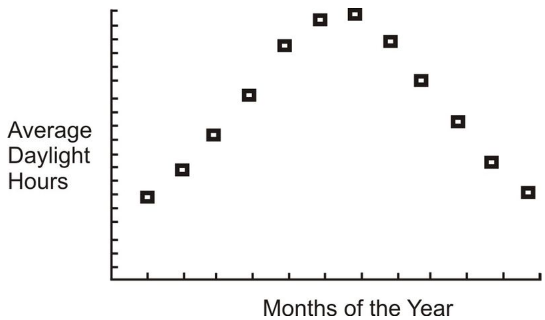


and



The $x-$ intercepts represent the break-even points of the company. The company must sell at least 2.25 units to cover any initial costs but when 177.7 units are sold, it no longer makes a profit.

9. a) Using the TI – 83 to create a scatter plot of the given data:



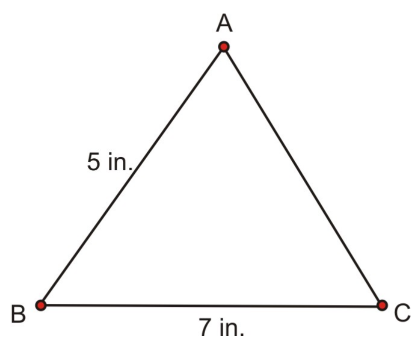
b) The period is twelve months.

c) The number of daylight hours in other areas would not show as much variance, so the amplitude of the graph would be smaller.

Angles in Triangles

Review Exercises:

1.



$\triangle ABC$ is isosceles. An isosceles triangle has two sides equal in length. Therefore AC is either 5 inches in length or 7 inches in length.

2. An obtuse triangle is one that has one angle that measures greater than 90° . A right triangle is one that has one angle that measures 90° . The sum of the angles of a triangle is 180° . Therefore a right triangle has one angle of 90° and two acute angles. A right triangle cannot be an obtuse triangle.

3. In any triangle, the sum of the three angles is 180° .

$$\begin{aligned}
\angle 1+\angle 2+\angle 3 &= 180^\circ \\
48^\circ+28^\circ+\angle &= 180^\circ \\
76^\circ+\angle 3 &= 180^\circ \\
\angle 3 &= 180^\circ-76^\circ \\
\angle 3 &= 104^\circ
\end{aligned}$$

4. a)

$$
\begin{aligned}
\angle 1+\angle 2+\angle 3 &= 180^\circ \\
90^\circ+\angle 2+\angle 3 &= 180^\circ \\
\angle 2+\angle 3 &= 180^\circ-90^\circ \\
\angle 2+\angle 3 &= 90^\circ
\end{aligned}
$$

Complementary Angles are two angles whose sum equals 90° . Therefore, the two acute angles of a right triangle are complementary angles.

b)

$$
\begin{aligned}
\angle 1 = 90^\circ \rightarrow & \text{ given} \\
\angle 2+\angle 3 &= 90^\circ \\
23^\circ+\angle 3 &= 90^\circ \\
\angle 3 &= 90^\circ-23^\circ \\
\angle 3 &= 67^\circ
\end{aligned}
$$

5. Let x represent $\angle D$. $\angle O = 2x$ since the measure of $\angle O$ is twice the measure of $\angle D$ and $\angle G = 3x$ since the measure of $\angle G$ is three times the measure of $\angle D$.

$$
\begin{aligned}
\text{Therefore:} x+2x+3x &= 180^\circ \\
6x &= 180^\circ \\
\frac{6x}{6} &= \frac{180^\circ}{6} \\
x &= 30^\circ
\end{aligned}
$$

$$
\begin{aligned}
\angle D &= x = 30^\circ \\
\angle O &= 2x = 60^\circ \\
\angle G &= 3x = 90^\circ
\end{aligned}
$$

6.

$$
\begin{aligned}
\frac{\overline{BC}}{\overline{EF}} &= \frac{\overline{AC}}{\overline{DF}} \\
\frac{8}{6} &= \frac{10}{\overline{DF}} \\
8\overline{DF} &= 60 \\
\frac{8\overline{DF}}{8} &= \frac{60}{8} \\
\frac{8}{DF} &= 7.5
\end{aligned}
$$

7. If two triangles are similar, then the corresponding angles are congruent. Therefore, $\angle B = \angle E$. In $\triangle ABC, \angle A = 30^\circ$ *and* $\angle C = 20^\circ$.

$$\angle B = 180^\circ - (30^\circ + 20^\circ)$$
$$\angle B = 130^\circ$$
$$\therefore \angle E = 130^\circ$$

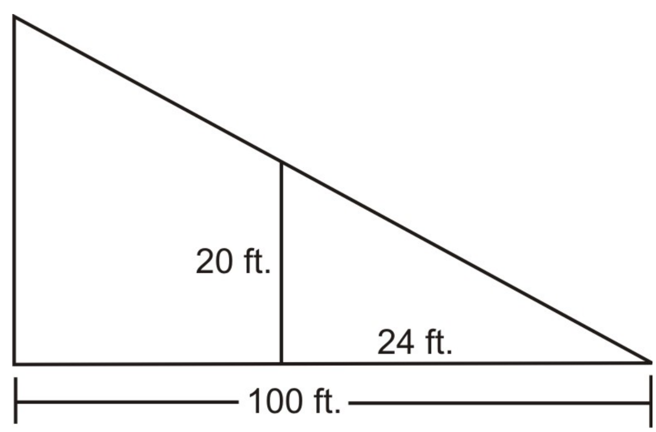
8. a)

$$\frac{\overline{AT}}{\overline{CT}} = \frac{\overline{OG}}{\overline{DG}}$$
$$\frac{8}{12} = \frac{6}{8}$$
$$\frac{2}{3} \neq \frac{3}{4} \qquad \triangle ACT \text{ and } \triangle DOG \text{ are not similar.}$$

b)

$$\frac{\overline{AB}}{\overline{DE}} = \frac{\overline{BC}}{\overline{EF}} = \frac{\overline{AC}}{\overline{DF}}$$
$$\frac{13}{6.5} = \frac{12}{6} = \frac{5}{2.5}$$
$$2 = 2 = 2 \qquad \triangle ABC \text{ and } \triangle DEF \text{ are similar.}$$

9.



$$\frac{20}{x} = \frac{24}{100}$$
$$24x = 2000$$
$$\frac{24x}{24} = \frac{2000}{24}$$
$$x = 83\frac{1}{3} \text{ feet}$$

The height of the building is $83\frac{1}{3}$ feet .

10. The answers to this question will vary. However, the answer should include the fact that corresponding sides of similar triangles are proportional and that corresponding angles are congruent.

Measuring Rotation

Review Exercises:

1. a) This angle is less than 90° and is an acute angle.

b) This angle is a rotation of 180° and is a straight angle.

2. a) The measure of this angle is greater than 90° but less than 180° . Since the terminal arm of the angle is less than half way between 90° and 180° , the approximate measure of the angle is 120° . A protractor could be used to determine the exact measure of the angle.

3. a) 85.5° expressed in degrees, minutes and seconds would be $85^\circ 30'$

$$\begin{aligned}
\frac{50}{100} &= \frac{x}{60}\\
100x &= 3000\\
\frac{100x}{100} &= \frac{3000}{100}\\
x &= 30
\end{aligned}$$

b) 12.15° expressed in degrees, minutes and seconds would be $12^\circ 9'$.

$$\begin{aligned}
\frac{15}{100} &= \frac{x}{60}\\
100x &= 900\\
\frac{100x}{100} &= \frac{900}{100}\\
x &= 9
\end{aligned}$$

c) 114.96° expressed in degrees, minutes and seconds would be $114^\circ 57' 3.6$”

$$\begin{aligned}
\frac{96}{100} &= \frac{x}{60}\\
100x &= 5760\\
\frac{100x}{100} &= \frac{5760}{100}\\
x &= 67.6
\end{aligned}
\qquad\qquad
\begin{aligned}
\frac{0.6}{60} &= \frac{s}{360}\\
60s &= 216\\
\frac{60s}{60} &= \frac{216}{60}\\
s &= 3.6
\end{aligned}$$

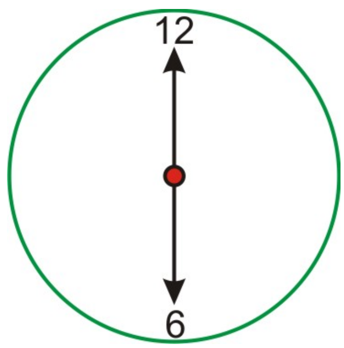
4. a)

$$\begin{aligned}
&54^\circ + \frac{10}{60} + \frac{25}{360}\\
&54^\circ + \frac{60}{360} + \frac{25}{360}\\
&54^\circ + \frac{85}{360}\\
&54^\circ + 0.236\\
&\approx 54.236^\circ
\end{aligned}$$

b)

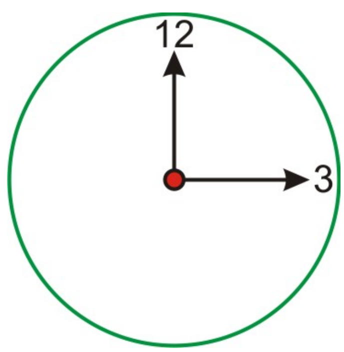
$$
\begin{aligned}
&17^\circ+\frac{40}{60}+\frac{5}{360}\\
&17^\circ+\frac{240}{360}+\frac{5}{360}\\
&17^\circ+\frac{245}{360}\\
&17^\circ+0.681\\
&\approx 17.681^\circ
\end{aligned}
$$

5. a)



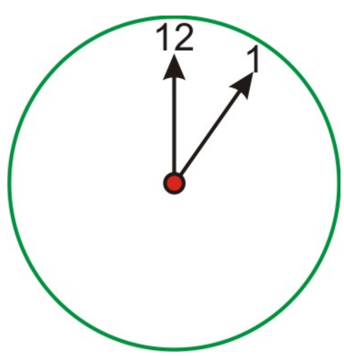
The angle between the hands of the clock at 6:00 is 180° .

b)



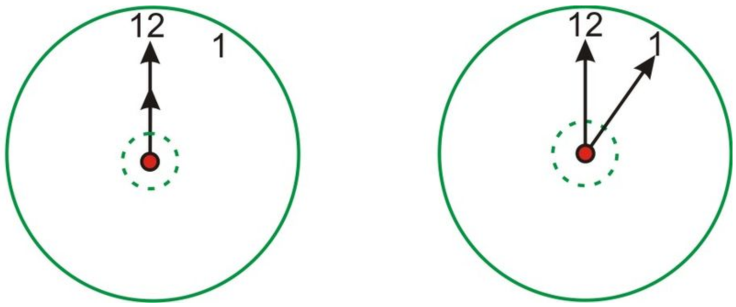
The angle between the hands of the clock at 3:00 is 90° .

c)



The angle between the hands of the clock at 1:00 is 30° .

6.



Between 12:00 and 1:00 o'clock, the arms of the clock rotate through an angle of 360° .

7.

$$C_{\text{inner track}} = \frac{1}{4}\pi d$$
$$C_{\text{inner track}} = \frac{1}{4}(\pi)(200\text{ m})$$
$$C_{\text{inner track}} = 50(\pi)\text{meters}$$

$$C_{\text{outer wheel}} = \pi d$$
$$C_{\text{outer wheel}} = (\pi)(0.6\text{m})$$
$$C_{\text{outer wheel}} = 0.6(\pi)\text{meters}$$

$$C_{\text{outer track}} = \frac{1}{4}\ \pi d$$
$$C_{\text{outer track}} = \frac{1}{4}((\pi)((204\text{ m})$$
$$C_{\text{outer track}} = 51(\pi)\text{ meters}$$

$$C_{\text{inner wheel}} = \pi d$$
$$C_{\text{oinner wheel}} = (\pi)(0.6\text{ m})$$
$$C_{\text{inner wheel}} = 0.6(\pi)\text{ meters}$$

$$C_{\text{outer track}} - C_{\text{inner track}} = 1\pi \qquad \text{and} \qquad \frac{1\pi}{0.6\pi} \approx 1.66666 \approx \frac{5}{3}$$

8. There are many answers to this question. The angles that are co-terminal with an angle of 90° can be expressed as $x = 90° + 360°k, k\varepsilon I$ where k is any integer.

Some examples of the co-terminal angles are

$$x = 90^\circ + 360^\circ = 450^\circ$$
$$x = 90^\circ + 720^\circ = 810^\circ$$
$$x = 90^\circ - 360^\circ = -270^\circ$$
$$x = 90^\circ - 720^\circ = -630^\circ$$

9. a) There are many answers to this question. The negative angles that are co-terminal with an angle of 120° can be expressed as $x = 120^\circ + 360^\circ k, k\varepsilon I$ where k is a negative integer. Some examples of the co-terminal angles of 120° that are negative angles are:

$$x = 120^\circ - 360^\circ = -240^\circ$$
$$x = 120^\circ - 720^\circ = -600^\circ$$

b) There are many answers to this question. The angles that are greater than 360° and co-terminal with an angle of 120° can be expressed as $x = 120^\circ + 360^\circ k, k\varepsilon I$ where k is a positive integer. Some examples of the co-terminal angles of 120° that are greater than 360° are:

$$x = 120^\circ + 360^\circ = 480^\circ$$
$$x = 120^\circ + 720^\circ = 840^\circ$$

10.

$$C_{\text{track}} = \frac{1}{2}\pi d$$
$$C_{\text{track}} = \frac{1}{2}(\pi)(240 \text{ m})$$
$$C_{\text{track}} = 120\pi \text{ meters(Inside Distance)}$$

The front outside wheel will complete the most rotations.

$$\frac{C_{\text{track}}}{C_{\text{front wheel}}} = \frac{122\pi}{0.6\pi} \approx 203 \text{ rotations}$$

$$C_{\text{track}} = \frac{1}{2}\pi d$$
$$C_{\text{track}} = \frac{1}{2}(\pi)((244 \text{ m})$$
$$C_{\text{track}} = 122(\pi)\text{meters}$$

$$C_{\text{front wheel}} = \pi d$$
$$C_{\text{front wheel}} = ((\pi)((0.6\text{m})$$
$$C_{\text{front wheel}} = 0.6(\pi) \text{ meters}$$

$$C_{\text{back wheel}} = \pi d$$
$$C_{\text{back wheel}} = ((\pi)(1.88\text{m})$$
$$C_{\text{back wheel}} = 1.88(\pi) \text{ meters}$$

The back inside wheel will complete the least number of rotations

$$\frac{C_{\text{track}}}{C_{\text{back wheel}}} = \frac{120\pi}{1.8\pi} \approx \frac{200}{3} \approx 67 \text{ rotations}$$

$$\text{degrees front tire} = 203 \ \text{ rotations} \times 360^\circ$$
$$\text{degrees front tire} = 73080^\circ$$

$$\text{degrees back tire} = 67 \text{ rotations} \times 360^\circ$$
$$\text{degrees back tire} = 24120^\circ$$

$$\text{degrees difference} = 73080^\circ - 24120^\circ$$
$$\text{degrees difference} = 48960^\circ$$

Defining Trigonometric Functions

Review Exercises:

1. In $\triangle ABC$, with respect to $\angle A$, the opposite side is 9 , the adjacent side is 12 , and the hypotenuse is 15 . The values of the six trigonometric functions for $\angle A$ are:

TABLE 1.1:

Function	Ratio	Value
$\sin \angle A$	$\frac{opp}{hyp}$	$\frac{9}{15} = \frac{3}{5}$
$\cos \angle A$	$\frac{adj}{hyp}$	$\frac{12}{15} = \frac{4}{5}$
$\tan \angle A$	$\frac{opp}{adj}$	$\frac{9}{12} = \frac{3}{4}$
$\csc \angle A$	$\frac{hyp}{opp}$	$\frac{15}{9} = \frac{5}{3}$
$\sec \angle A$	$\frac{hyp}{adj}$	$\frac{15}{12} = \frac{5}{4}$
$\cot \angle A$	$\frac{adj}{opp}$	$\frac{12}{9} = \frac{4}{3}$

2. a) In $\triangle VET$ the hypotenuse is:

$$(h)^2 = (s_1)^2 + (s_2)^2$$
$$(h)^2 = (8)^2 + (15)^2$$
$$(h)^2 = 64 + 225$$
$$\sqrt{h^2} = \sqrt{289} \quad h = 17$$

b) In $\triangle VET$, with respect to T , the opposite side is 15 , the adjacent side is 8 , and the hypotenuse is 17 . The values of the six trigonometric functions for $\angle T$ are:

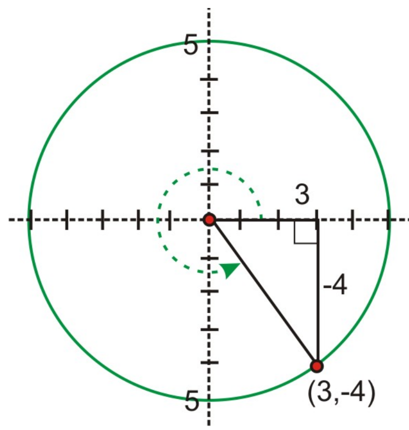
TABLE 1.2:

Function	Ratio	Value
$\sin \angle T$	$\frac{opp}{hyp}$	$\frac{15}{17}$
$\cos \angle T$	$\frac{adj}{hyp}$	$\frac{8}{17}$

TABLE 1.2: (continued)

Function	Ratio	Value
$\tan\angle T$	$\frac{opp}{adj}$	$\frac{15}{8}$
$\csc\angle T$	$\frac{hyp}{opp}$	$\frac{17}{15}$
$\sec\angle T$	$\frac{hyp}{adj}$	$\frac{17}{8}$
$\cot\angle T$	$\frac{adj}{opp}$	$\frac{8}{15}$

3. a)



$$\text{The radius of the circle is } (h)^2 = (s_1)^2 + (s_2)^2$$
$$(h)^2 = (3)^2 + (-4)^2$$
$$(h)^2 = 9 + 16$$
$$\sqrt{h^2} = \sqrt{25} \quad \therefore h = 5$$

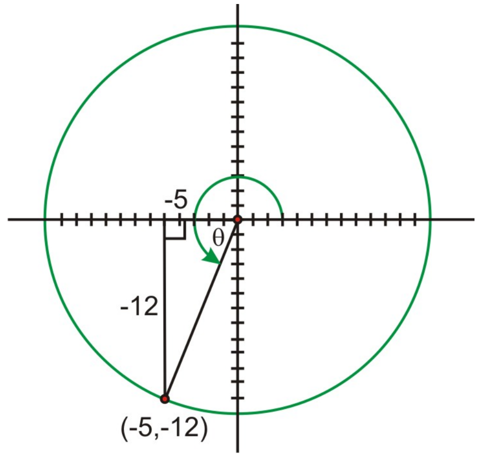
With respect to the angle in standard position, θ , the hypotenuse is 5 , the opposite is -4 , and the adjacent is 3 .

b)

TABLE 1.3:

Function	Ratio	Value
$\sin\theta$	$\frac{opp}{hyp}$	$-\frac{4}{5}$
$\cos\theta$	$\frac{adj}{hyp}$	$\frac{3}{5}$
$\tan\theta$	$\frac{opp}{adj}$	$-\frac{4}{3}$
$\csc\theta$	$\frac{hyp}{opp}$	$-\frac{5}{4}$
$\sec\theta$	$\frac{hyp}{adj}$	$\frac{5}{3}$
$\cot\theta$	$\frac{adj}{opp}$	$-\frac{3}{4}$

4. a)



The radius of the circle is

$$(h)^2 = (s_1)^2 + (s_2)^2$$
$$(h)^2 = (-5)^2 + (-12)^2$$
$$(h)^2 = 25 + 144$$
$$(h)^2 = \sqrt{169} \quad \therefore h = 13$$

With respect to the angle in standard position, θ , the hypotenuse is 13 , the opposite is -12 , and the adjacent is -5
.

b)

TABLE 1.4:

Function	Ratio	Value
$\sin\theta$	$\frac{opp}{hyp}$	$-\frac{12}{13}$
$\cos\theta$	$\frac{adj}{hyp}$	$-\frac{5}{13}$
$\tan\theta$	$\frac{opp}{adj}$	$\frac{-12}{-5} = \frac{12}{5}$
$\csc\theta$	$\frac{hyp}{opp}$	$-\frac{13}{12}$
$\sec\theta$	$\frac{hyp}{adj}$	$-\frac{13}{5}$
$\cot\theta$	$\frac{adj}{opp}$	$\frac{-5}{-12} = \frac{5}{12}$

5.

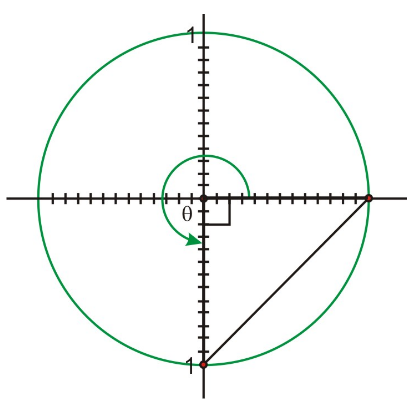


TABLE 1.5:

Function	Value
$\sin\theta$	-1
$\cos\theta$	0
$tan\theta$	undefined
$\csc\theta$	-1
$\sec\theta$	undefined
$\cot\theta$	0

6. a) The measure of $\angle DAB$ is 60° which is the sum of $\angle BAC\ and\ \angle DAC$. The measure of each angle in $\triangle DAB$ is 60° . Therefore the triangle is equiangular.

b) The measure of the side BD of $\triangle DAB$ is 1 because it is the third side of $\triangle DAB$ which is also an equilateral triangle.

c) The measure of BC and CD is $\frac{1}{2}$ The altitude AC of the equilateral triangle bisects the base BD which has a length of one.

d) The ordered pair can be obtained by first using the Pythagorean Theorem to determine the measure of AC .

$$\begin{aligned}(h)^2 &= (s_1)^2 + (s_2)^2\\ (1)^2 &= (-.5)^2 + (s)^2\\ 1 &= .25 + s^2\\ 1 - 0.25 &= s^2\\ \sqrt{0.75} &= \sqrt{s^2} && \therefore s = 0.8660 \text{ which is equivalent to}\\ \frac{\sqrt{3}}{2}.\end{aligned}$$

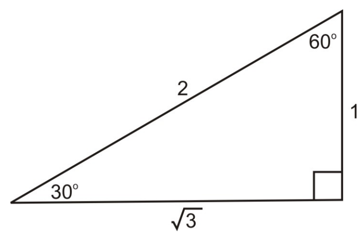
If $\angle BAC$ were represented as an angle in standard position, the coordinates on the unit circle would be $(\cos 30^\circ, \sin 30^\circ)$ or $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$.

e) If $\angle ABC$ were represented as an angle in standard position, the opposite side would be $\sqrt{3}$, and the adjacent side would be 1 . Therefore the coordinates on the unit circle would be $(\cos 60^\circ, \sin 60^\circ)$ or $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$.

7.

$$
\begin{aligned}
(h)^2 &= (s_1)^2 + (s_2)^2 \\
(1)^2 &= (n)^2 + (n)^2 \\
1 &= n^2 + n^2 \\
1 &= 2n^2 \\
\frac{1}{2} &= \frac{2n^2}{2} \\
\pm\sqrt{\frac{1}{2}} &= \sqrt{n^2} \qquad \therefore n = \pm\frac{1}{\sqrt{2}} \text{ Rationalize the denominator} \\
\pm\frac{1}{\sqrt{2}}\left(\frac{\sqrt{2}}{\sqrt{2}}\right) &= \pm\frac{\sqrt{2}}{\sqrt{4}} = \pm\frac{\sqrt{2}}{2} \\
n &= \frac{\sqrt{2}}{2} \qquad \text{The angle is in the first quadrant so the values of (x, y) are positive}
\end{aligned}
$$

8.

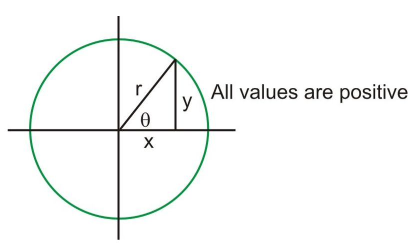


To determine the values of the six trigonometric functions for 60° , the following special triangle may be used.

TABLE 1.6:

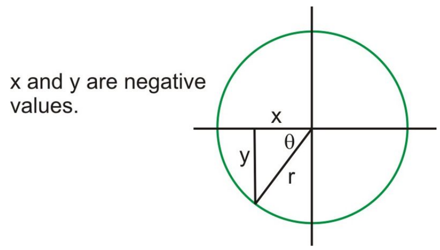
Function	Ratio	Value
$\sin 60^\circ$	$\frac{opp}{hyp}$	$\frac{\sqrt{3}}{2}$
$\cos 60^\circ$	$\frac{adj}{hyp}$	$\frac{1}{2}$
$\tan 60^\circ$	$\frac{opp}{adj}$	$\frac{\sqrt{3}}{1}$
$\csc 60^\circ$	$\frac{hyp}{opp}$	$\frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$
$\sec 60^\circ$	$\frac{hyp}{adj}$	$\frac{2}{1}$
$\cot 60^\circ$	$\frac{adj}{opp}$	$\frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$

9. An angle in standard position in the first quadrant



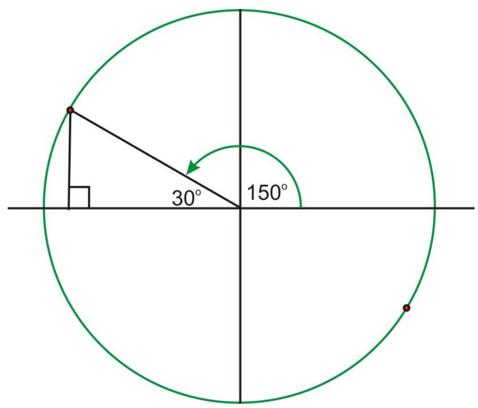
$\tan\theta = \frac{oppy}{adj(x)}$ Since both x and y are positive quantities, then the function will also be positive.

An angle in standard position in the third quadrant:



$\tan\theta = \frac{opp(y)}{adj(x)}$ Since both x and y are negative quantities, then the function will be positive.

10. An angle of 150° drawn in standard position is equivalent to a reference angle of 30° drawn in the second quadrant.



The coordinates of this angle on the unit circle are $(\cos 30^\circ, \sin 30^\circ)$ which would be $\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$

Trigonometric Functions of Any Angle

Review Exercises:

1. The reference angle for each of the following angles is:

a) 190° 190° [U+0080][U+0093] $180^\circ = 10^\circ$

b) [U+0080][U+0093] 60° 360° [U+0080][U+0093] $300^\circ = 60^\circ$ A negative angle indicates that the angle opens clockwise.

c) 1470° 1470° [U+0080][U+0093] $4(360^\circ) = 30^\circ$

d) [U+0080][U+0093] 135° 225° [U+0080][U+0093] $180^\circ = 45^\circ$

2. The coordinates for each of the following angles are:

a)

300°	The reference angle is 360° [U+0080][U+0093] $300^\circ = 60^\circ$ (4^{th} quadrant)
	$(\cos 60^\circ, \sin 60^\circ) = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$

b)

-150° The reference angle is 180° [U+0080] [U+0093] $150^\circ = 30^\circ$ (3^{rd} quadrant)

$$(\cos 30^\circ, \sin 30^\circ) = \left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$$

c)

405° The reference angle is 405° [U+0080] [U+0093] $360^\circ = 45^\circ$ (1^{st} quadrant)

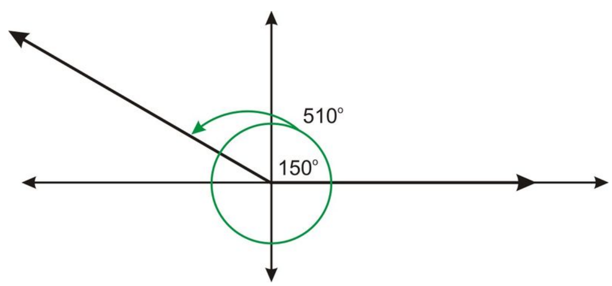
$$(\cos 45^\circ, \sin 45^\circ) = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$$

3. a) $\sin 210^\circ$ is equivalent to $\sin 30^\circ$ in the 3^{rd} quadrant. Its value is $-\frac{1}{2}$.

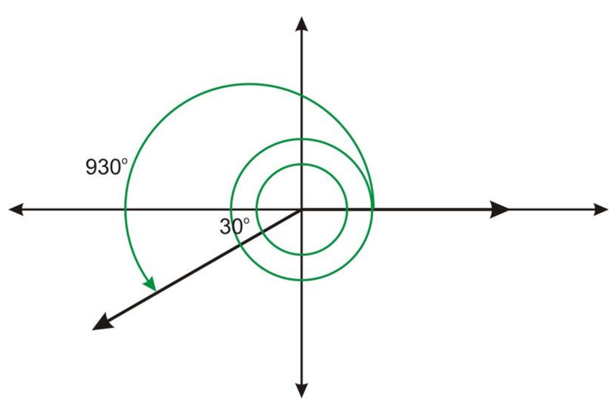
b) $\tan 270^\circ$ is equivalent to $\tan 90^\circ$. Its value is undefined.

c) $\csc 120^\circ$ is equivalent to $\csc 60^\circ$ in the 2^{nd} quadrant. Cosecant is the reciprocal of sine so the value will be positive. Its value is $\frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$.

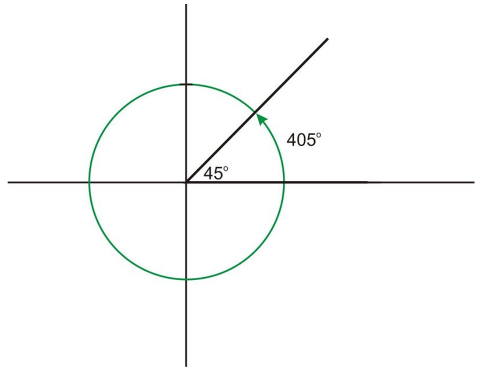
4. a) An angle of 510° has a reference angle of 30° in the 2^{nd} quadrant. Therefore, the value of $\sin 510^\circ$ is $\frac{1}{2}$.



b) An angle of 930° has a reference angle of 30° in the 3^{rd} quadrant. Therefore, the value of $\cos 930^\circ$ is $-\frac{\sqrt{3}}{2}$.



c) An angle of 405° has a reference angle of 45° in the 1^{st} quadrant. The value of $\csc 405^\circ$ is $\frac{\sqrt{2}}{1}$.



5. a) An angle of [U+0080][U+0093]150° has a reference angle of 30° in the 3rd quadrant. Therefore the value of $\cos(-150^\circ)$ is $-\frac{\sqrt{3}}{2}$.

b) An angle of [U+0080][U+0093]45° has a reference angle of 45° in the 4th quadrant. Therefore the value of $\tan(-45^\circ)$ is -1 .

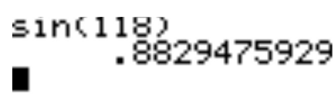
c) An angle of [U+0080][U+0093]240° has a reference angle of 60° in the 2nd quadrant. Therefore the value of $\sin(-240^\circ)$ is $\frac{\sqrt{3}}{2}$.

6. Using the table in the lesson the value of $\cos 100^\circ$ is approximately -0.1736 .

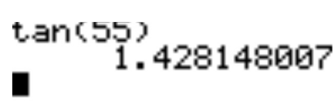
7. Using the table in the lesson, the angle that has a sine value of 0.2 is between 165° and 170° .

8. The tangent of 50° is approximately 1.1918 and this value is very reasonable because $\tan 45^\circ$ is 1 . As the measure of the angle gets larger so does the tangent value of the angle.

9. a) The value of $\sin 118^\circ$ using the calculator is approximately $\sin 118^\circ \approx .8829$



b) The value of $\tan 55^\circ$ using the calculator is approximately $\tan 55^\circ \approx 1.4281$.



10. From observing the value displayed in the table, the conjecture that can be made is $\sin(a)+\sin(b) \neq \sin(a+b)$.

11. This area represents a worksheet for $\sin(a)$ and $(sina)^2$.

$$\begin{aligned}
\sin 0^\circ &= 0 & (0)^2 &= 0\\
\sin 25^\circ &= 0.4226 & (0.4226)^2 &= 0.1786\\
\sin 45^\circ &= \frac{\sqrt{2}}{2} & \left(\frac{\sqrt{2}}{2}\right)^2 &= \frac{1}{2}\\
\sin 80^\circ &= 0.9848 & (0.9848)^2 &= 0.9698\\
\sin 90^\circ &= 1 & (1)^2 &= 1\\
\sin 120^\circ &= \frac{\sqrt{3}}{2} & \left(\frac{\sqrt{3}}{2}\right)^2 &= \frac{3}{4}\\
\sin 235^\circ &= -0.8192 & (-0.8192)^2 &= 0.6711\\
\sin 310^\circ &= -0.7660 & (-0.7660)^2 &= 0.5868
\end{aligned}$$

This area represents a worksheet for $\cos(a)$ and $(\cos a)^2$

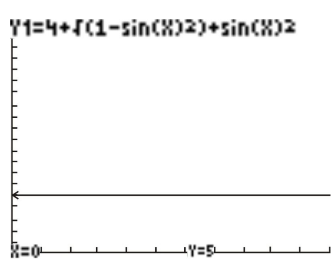
$$\begin{aligned}
\cos 0^\circ &= 1 & (1)^2 &= 1\\
\cos 25^\circ &= 0.9063 & (0.9063)^2 &= 0.8214\\
\cos 45^\circ &= \frac{\sqrt{2}}{2} & \left(\frac{\sqrt{2}}{2}\right)^2 &= \frac{1}{2}\\
\cos 80^\circ &= 0.1736 & (0.1736)^2 &= 0.0301\\
\cos 90^\circ &= 0 & (0)^2 &= 0\\
\cos 120^\circ &= -\frac{1}{2} & \left(-\frac{1}{2}\right)^2 &= \frac{1}{4}\\
\cos 235^\circ &= -0.5736 & (-0.5736)2 &= 0.3290\\
\cos 310^\circ &= 0.6428 & (0.6428)^2 &= 0.4132
\end{aligned}$$

From the above results the following conjecture can be made:

$$(\sin a)^2 + (\cos a)^2 = 1$$

12. $g(x) = 4 + \sqrt{1-\sin^2 x} + \sin^2 x$ The conjecture that would be made about the value of this function is that it would equal 5 .

Using the TI-83 to graph the function:



In order for this to occur with the above function $1-\sin^2 x$ should be changed to $(\cos^2 x)^2$ to result in $\cos^2 x + \sin^2 x$ which equals one.

Relating Trigonometric Functions

Review Exercise: Pages 80 – 82

1. a)

$$\sec\theta = 4 \qquad \cos\theta = \frac{1}{\sec\theta} \qquad \therefore \cos\theta = \frac{1}{4}$$

b)

$$\sin\theta = \frac{1}{3} \qquad \csc\theta = \frac{1}{\sin\theta} \qquad \csc\theta = \frac{1}{\frac{1}{3}} \qquad \therefore \csc\theta = 3$$

2. a)

TABLE 1.7:

Angle	Sin	Csc
10	0.1736	5.7604
5	0.0872	11.4737
1	0.0175	57.2987
0.5	0.0087	114.5930
0.1	0.0017	572.9581
0	0	*undefined*
−0.1	−0.0017	−572.9581
−0.5	−0.0087	−114.5930
−1	−0.0175	−57.2987
−5	−0.0872	−11.4737
−10	−0.1736	−5.7604

b) As the measure of the angle approaches zero degrees, the values of the cosecant increase greatly.

c) The value of the sine function has a maximum of one. However, the cosecant function has no maximum value. Its value continues to increase.

d) The range of the cosecant function has no values between -1 and $+1$. However, it does have values from -1 to $-\infty$ and from $+1$ to $+\infty$.

3. Any angles that resulted in a value of zero for the cosine of the angle are excluded from the domain of the secant function. These angles include $90^\circ, 270^\circ, 450^\circ$, etc.

4. To answer this question correctly, the following diagram that shows in which quadrant the trigonometric functions are positive, will be used to determine the sign of the given function.

S	A
Sine Cosecant	All
Tangent Cotangent	Cosine Secant
T	C

a) $\sin 80^\circ \rightarrow$ The angle is located in the 1st quadrant and its value will be positive.

b) $\cos 200^\circ \rightarrow$ The angle is located in the 3rd quadrant and its value will be negative.

c) $\cot 325^\circ \rightarrow$ The angle is located in the 4th quadrant and its value will be negative.

d) $\tan 110^\circ \rightarrow$ The angle is located in the 2nd quadrant and its value will be negative.

5.

$$\cos\theta = \frac{adj}{hyp} = \frac{6}{10}; \qquad \sin\theta = \frac{opp}{hyp} = \frac{8}{10}; \qquad \tan\theta = \frac{opp}{adj} = \frac{8}{6} = \frac{4}{3}$$

6. In the 3rd quadrant, both the sine function and the cosine function have negative values. $\tan\theta = \frac{\sin\theta}{\cos\theta}$ and $\cot\theta = \frac{\cos\theta}{\sin\theta}$. The result of dividing two negative values is positive. Therefore, in the 3rd quadrant these quotient identities will have a positive value.

7. All angles in the 1st quadrant have a positive value.

$$\begin{aligned} \sin\theta &= 0.4 \\ \sin^{-1}(\sin\theta) &= \sin^{-1}(0.4) \\ \theta &\approx 23.58^\circ \end{aligned}$$

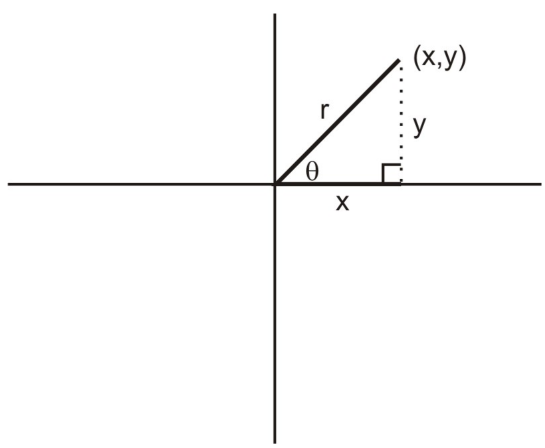
Therefore $\cos 23.58^\circ \approx 0.9165$.

8. All angles in the 1st quadrant have a positive value. If $\cot\theta = 2$ then $\tan\theta = \frac{1}{2}$

$$\begin{aligned} \tan^{-1}(\tan\theta) &= \tan^{-1}\left(\frac{1}{2}\right) \\ \theta &\approx 26.57^\circ \end{aligned}$$

Therefore $\csc 26.57^\circ \approx 2.2357$ (Note: $\csc\theta = \frac{1}{\sin\theta}$).

9.



From the above diagram, $\sin\theta = \frac{y}{r}; \cos\theta = \frac{x}{r}$ and $x^2 + y^2 = r^2$

$$\begin{aligned} & x^2 + y^2 = r^2 \\ & \frac{x^2}{y^2} + \frac{y^2}{r^2} = \frac{r^2}{r^2} && \text{Dividing through by } r^2 \\ & \cos^2\theta + \sin^2\theta = 1 && \text{Replacing the ratios with the correct functions as defined above.} \end{aligned}$$

The Pythagorean Identity $\cos^2\theta + \sin^2\theta = 1$ can now be used to prove $1 + \tan^2\theta = \sec^2\theta$.

Proof:

$$\begin{aligned}
\cos^2\theta + \sin^2\theta &= 1 \\
\frac{\cos^2\theta}{\cos^2\theta} + \frac{\sin^2\theta}{\cos^2\theta} &= \frac{1}{\cos^2\theta} && \text{Dividing through by } \cos^2\theta \\
1 + \tan^2\theta &= \sec^2\theta && \text{Using identities for substitutions:. } \tan\theta = \frac{\sin\theta}{\cos\theta} \text{ and } \sec\theta = \frac{1}{\cos\theta}
\end{aligned}$$

10. It is necessary to indicate the quadrant in which the angle is located in order to determine the correct angle. When using the Pythagorean Identities, the equations are quadratic and a quadratic equation has two possible solutions. If the quadrant is stated in the question, then only one answer is acceptable.

Applications of Right Triangle Trigonometry

Review Exercises:

1. To solve a triangle means to determine the measurement of all angles and all sides of the given triangle. In $\triangle ABC$:

$$\begin{aligned}
a &\approx 9.33 & \angle A &= 58^\circ & \angle A &= 180^\circ - (32^\circ + 90^\circ) \\
b &\approx 5.83 & \angle B &= 32^\circ & \angle A &= 58^\circ \\
c &= 11 & \angle C &= 90^\circ
\end{aligned}$$

$$\begin{aligned}
\sin B &= \frac{opp}{hyp} & \cos B &= \frac{adj}{hyp} \\
\sin 32^\circ &= \frac{b}{11} & \cos 32^\circ &= \frac{a}{11} \\
0.5299 &= \frac{b}{11} & 0.8480 &= \frac{a}{11} \\
(11)(0.5299) &= (11)\left(\frac{b}{11}\right) & (11)(0.8480) &= (11)\left(\frac{a}{11}\right) \\
5.83 &\approx b & 9.33 &\approx a
\end{aligned}$$

2. Anna is correct. In order to solve a triangle, the minimum amount of information that must be given is the measure of two angles and one side, or one angle and two sides.

3.

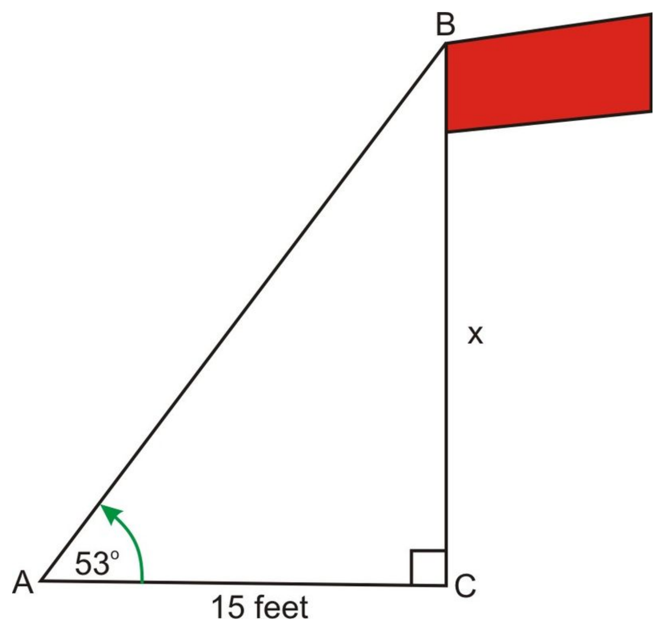
$$\begin{aligned}
(h)^2 &= (s_1)^2 + (s_2)^2 \\
(h)^2 &= (6)^2 + (5.03)^2 \\
\sqrt{h^2} &= \sqrt{61.3009} && \therefore h \approx 7.829 \approx 7.83 \text{ This answer confirms those given in example 2.}
\end{aligned}$$

4. $\sin B = \frac{3}{5} = 0.6 \quad \sin 30^\circ = \frac{1}{2} = 0.5$ Therefore, the measure of $\angle B$ is larger than 30° .

Using a calculator,

$$\begin{aligned}\sin^{-1}(\sin B) &= \sin^{-1}(0.6)\\ \angle B &\approx 36.87^\circ \approx 37^\circ\end{aligned}$$

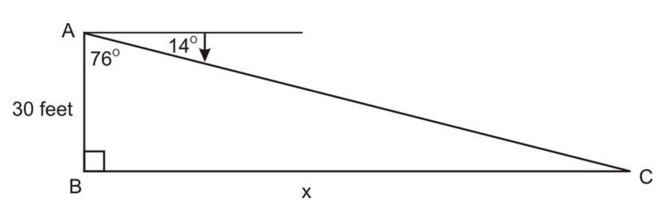
5.



$$\begin{aligned}\tan\angle A &= \frac{opp}{adj}\\ \tan 53^\circ &= \frac{x}{15}\\ 1.3270 &= \frac{x}{15}\\ (15)(1.3270) &= (15)\left(\frac{x}{15}\right)\\ 19.91 \text{ feet} &\approx x\end{aligned}$$

The length of the flagpole is approximately 19.9 feet.

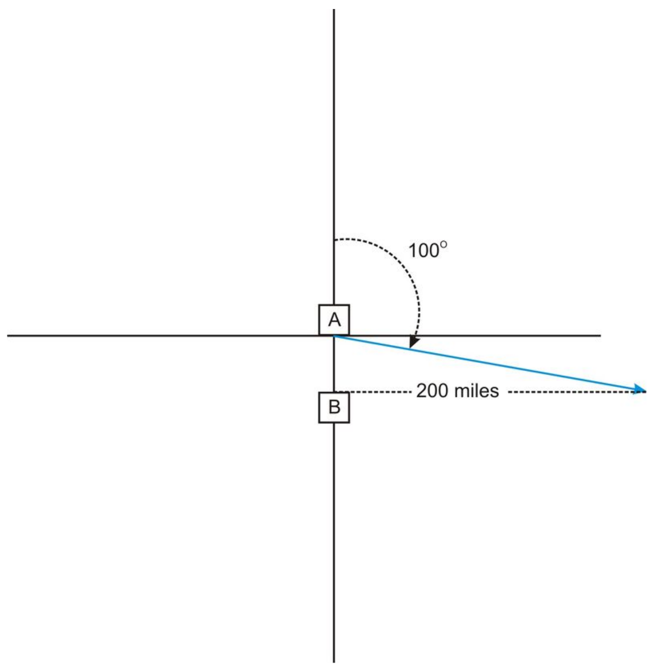
6.



$$\begin{aligned}
\tan\angle BAC &= \frac{opp}{adj} \\
\tan 76^\circ &= \frac{x}{30} \\
4.0108 &= \frac{x}{30} \\
(30)(4.0108) &= (30)\left(\frac{x}{30}\right) \\
120.32 \text{ feet} &\approx x
\end{aligned}$$

The house is approximately 120.3 feet away.

7.



$$\begin{aligned}
\sin A &= \frac{opp}{hyp} \\
\sin 80^\circ &= \frac{200}{x} \\
0.9848 &= \frac{200}{x} \\
(x)(0.9848) &= (x)\left(\frac{200}{x}\right) \\
0.9848x &= 200 \\
\frac{0.9848x}{0.9848} &= \frac{200}{0.9848} \\
x &\approx 203.09 \approx 203 \text{ miles}
\end{aligned}$$

$$\begin{aligned}
\tan\angle A &= \frac{opp}{adj} \\
\tan 80^\circ &= \frac{200}{x} \\
5.6713 &= \frac{200}{x} \\
(x)(5.6713) &= (x)\left(\frac{200}{x}\right) \\
5.6713x &= 200 \\
\frac{5.6713x}{5.6713} &= \frac{200}{5.6713} \\
x &\approx 35.27 \approx 35.3 \text{ miles}
\end{aligned}$$

The plane has traveled approximately 203 miles **.**

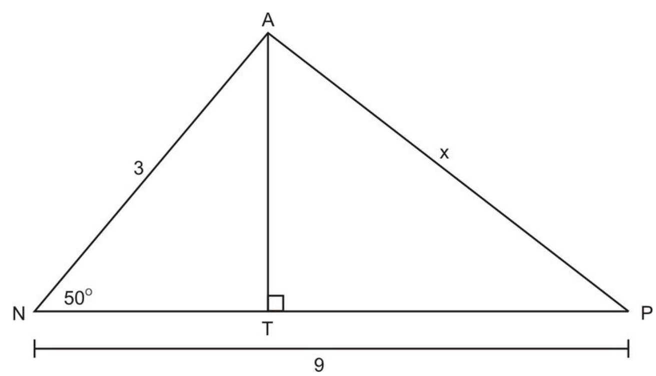
City A and City B are approximately 35.3 miles **apart.**

8.

$$\tan \angle C = \frac{opp}{adj}$$
$$\tan 40^\circ = \frac{x}{50}$$
$$0.8391 = \frac{x}{50}$$
$$(50)(0.8391) = (50)\left(\frac{x}{50}\right)$$
$$x \approx 41.96 \text{ feet}$$

The lake is approximately 41.96 feet wide.

9.



$\triangle TAN$

$\sin N = \frac{opp}{hyp}$	$\cos \angle N = \frac{adj}{hyp}$
$\sin 50^\circ = \frac{AT}{3}$	$\cos 50^\circ = \frac{NT}{3}$
$0.7660 = \frac{AT}{3}$	$0.6428 = \frac{NT}{3}$
$(3)(0.7660) = (3)\left(\frac{AT}{3}\right)$	$(3)(0.6428) = (3)\left(\frac{NT}{3}\right)$
$2.29 \approx 2.3 \approx AT$	$1.93 \approx 1.9 \approx NT$

$\triangle PAT$

$\overline{PT} = \overline{NP} - \overline{NT}$	$(h)^2 = (s_1)^2 + (s_2)^2$
$\overline{PT} = 9.0 - 1.9$	$(h)^2 (7.1)^2 + (2.3)^2$
$\overline{PT} = 7.1$	$\sqrt{h^2} = \sqrt{55.7} \quad \therefore h \approx 7.46$
$\overline{AT} = 2.3$	The length of side x is approximately 7.46

Chapter 2 Circular Functions - Solution Key

Chapter Outline

2.1 Circular Functions

Radian Measure

Review Exercises

1.

a) The circle that is missing appears to be one-third of the circle. Therefore the measure of the angle could be estimated to be 120° .

b) $120^\circ \cdot \frac{\pi}{180^\circ} = \frac{120^\circ\pi}{180^\circ} = \frac{2\pi}{3}$ radians

c) The part of the cheese that remains has a measure of $360^\circ - 120^\circ = 240^\circ$.

$$240^\circ \cdot \frac{\pi}{180^\circ} = \frac{240^\circ\pi}{180^\circ} = \frac{4\pi}{3} \text{ radians}$$

2.

TABLE 2.1:

Angle in Degrees	Radian Measure
240°	$240^\circ \cdot \frac{\pi}{180^\circ} = \frac{240^\circ\pi}{180^\circ} = \frac{4\pi}{3}$ radians
270°	$270^\circ \cdot \frac{\pi}{180^\circ} = \frac{270^\circ\pi}{180^\circ} = \frac{3\pi}{2}$ radians
315°	$315^\circ \cdot \frac{\pi}{180^\circ} = \frac{315^\circ\pi}{180^\circ} = \frac{7\pi}{4}$ radians
-210°	$-210^\circ \cdot \frac{\pi}{180^\circ} = \frac{-210^\circ\pi}{180^\circ} = \frac{7\pi}{6}$ radians
120°	$120^\circ \cdot \frac{\pi}{180^\circ} = \frac{120^\circ\pi}{180^\circ} = \frac{2\pi}{3}$ radians
15°	$15^\circ \cdot \frac{\pi}{180^\circ} = \frac{15^\circ\pi}{180^\circ} = \frac{\pi}{12}$ radians
-450°	$-450^\circ \cdot \frac{\pi}{180^\circ} = \frac{-450^\circ\pi}{180^\circ} = -\frac{5\pi}{2}$ radians
72°	$72^\circ \cdot \frac{\pi}{180^\circ} = \frac{72^\circ\pi}{180^\circ} = \frac{2\pi}{5}$ radians
720°	$720^\circ \cdot \frac{\pi}{180^\circ} = \frac{720^\circ\pi}{180^\circ} = 4\pi$ radians
330°	$330^\circ \cdot \frac{\pi}{180^\circ} = \frac{330^\circ\pi}{180^\circ} = \frac{11\pi}{6}$ radians

3.

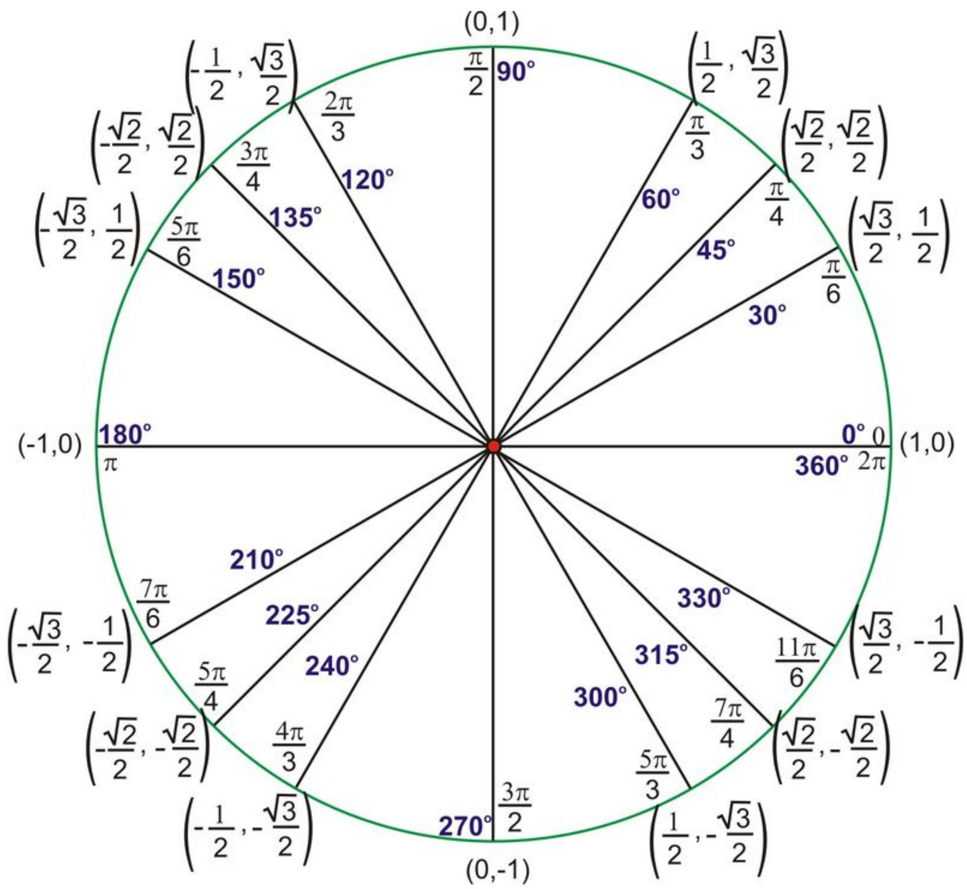
TABLE 2.2:

Angle in Radians	Degree Measure
$\frac{\pi}{2}$	$\frac{\cancel{\pi}}{2} \cdot \frac{180^\circ}{\cancel{\pi}} = \frac{180^\circ}{2} = 90^\circ$
$\frac{11\pi}{5}$	$\frac{\cancel{11}\pi}{5} \cdot \frac{180^\circ}{\cancel{\pi}} = \frac{1980^\circ}{5} = 396^\circ$
$\frac{2\pi}{3}$	$\frac{\cancel{2}\pi}{3} \cdot \frac{180^\circ}{\cancel{\pi}} = \frac{360^\circ}{3} = 120^\circ$
5π	$5\cancel{\pi} \cdot \frac{180^\circ}{\cancel{\pi}} = 900^\circ$
$\frac{7\pi}{2}$	$\frac{7\cancel{\pi}}{2} \cdot \frac{180^\circ}{\cancel{\pi}} = \frac{1260^\circ}{2} = 630^\circ$
$\frac{3\pi}{10}$	$\frac{3\cancel{\pi}}{10} \cdot \frac{180^\circ}{\cancel{\pi}} = \frac{540^\circ}{10} = 54^\circ$
$\frac{5\pi}{12}$	$\frac{5\cancel{\pi}}{12} \cdot \frac{180^\circ}{\cancel{\pi}} = \frac{900^\circ}{12} = 75^\circ$

TABLE 2.2: (continued)

Angle in Radians	Degree Measure
$-\frac{13\pi}{6}$	$-\frac{13\cancel{\pi}}{6}\frac{180^\circ}{\cancel{\pi}} = \frac{2340^\circ}{6} = -390^\circ$
8π	$8\cancel{\pi}\cdot\frac{180^\circ}{\cancel{\pi}} = 1440^\circ$
$\frac{4\pi}{15}$	$\frac{4\cancel{\pi}}{15}\frac{180^\circ}{\cancel{\pi}} = \frac{720^\circ}{15} = 48^\circ$

4.



5.

a) $\frac{6\pi}{7}\text{rad} = \frac{6(180^\circ)}{7} \approx 154.3^\circ$

b) $1\text{rad} = \frac{180^\circ}{\pi} \approx 57.3^\circ$

c) $3\text{rad} = 57.3^\circ.\ 3 \approx 171.9^\circ$

d) $\frac{20\pi}{11} = \frac{20(180^\circ)}{11} \approx 327.3^\circ$

6.

a) $\sin\ 210^\circ = -\frac{1}{2}$

b) Gina calculated $\sin\ 210$ wit her calculator in radian mode.

7.

TABLE 2.3:

Angle (x)	$\text{Sin}(x)$	$\text{Cos}(x)$	$\text{Tan}(x)$
$\frac{5\pi}{4}(225^\circ \to 45^\circ)$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	1
$\frac{11\pi}{6}(330^\circ \to 30^\circ)$	$-\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{3}$
$\frac{2\pi}{3}(120^\circ \to 60^\circ)$	$\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$-\sqrt{3}$
$\frac{\pi}{2}(90^\circ)$	1	0	*undefined*
$\frac{7\pi}{2}(630^\circ \to 270^\circ)$	-1	0	*undefined*

Applications of Radian Measure

Review Exercises

1.

a) $\frac{360^\circ}{24} = 15^\circ \qquad 15^\circ \cdot \frac{\pi}{180^\circ} = \frac{15\pi}{180} = \frac{\pi}{12}\text{rad}$

b) $\frac{\pi}{12} \approx 0.3\text{rad}$

c) 15°

2.

a) $\frac{360^\circ}{12} = 30^\circ \qquad 30^\circ\left(\frac{\pi}{180^\circ}\right) = \frac{30\pi}{180} = \frac{\pi}{6}\text{rad}$

b)

$$\frac{\pi}{6}(0.5\text{m}) \approx 0.262\text{m}$$
$$0.262\text{m} \cdot 100\text{cm/m} \approx 26\text{cm}$$

3.

a) $\frac{360^\circ}{32} = \frac{45^\circ}{4} \qquad \frac{45^\circ}{4}\left(\frac{\pi}{180^\circ}\right) = \frac{45\pi}{720} = \frac{\pi}{16}\text{rad}$

b) The distance between two consecutive dots on the circle is $\frac{\pi}{16}\text{rad}$. Since the chord spans 13 dots, the measure of the central angle is $\frac{13\pi}{16}\text{rad}$ The length of the chord is:

$$c = 2r\ \sin\frac{\theta}{2}$$
$$c = 2(1.20\text{m})\sin\frac{13\pi}{16}\left(\frac{1}{2}\right)$$
$$c = 2(1.20\text{m})\sin\frac{13\pi}{32}$$
$$c \approx 2.297 \approx 2.3\text{m}$$
$$c \approx 2.3\text{m}\,(100\text{cm/m}) \approx \ \text{cm}$$

4. a) $\frac{360^\circ}{12} = 30^\circ \qquad 30^\circ\left(\frac{\pi}{180^\circ}\right) = \frac{30\pi}{180} = \frac{\pi}{6}$ rad

The area of each designated is equal to the area of the outer sector – the area of the inner sector.

$$\begin{aligned}A_{\text{(outer)}}\frac{1}{2}r^2\theta - A_{\text{(inner)}}\frac{1}{2}r^2\theta\\ A_{\text{(outer)}}\frac{1}{2}(110)^2\left(\frac{\pi}{6}\right) - A_{\text{(inner)}}\frac{1}{2}(55)^2\left(\frac{\pi}{6}\right)\\ \approx 3167.77 - 791.94 \approx 2375.83 \approx 2376 \text{ ft}^2\end{aligned}$$

The approximate area of each section is 2376 ft^2 .

The students from Archimedes High school have four allotted sections:

$$4(2376 \text{ ft}^2) = 9504 \text{ ft}^2$$

b) There are three sections allotted for general admission:

$$3(2376 \text{ ft}^2) = 7128 \text{ ft}^2$$

c) The press and the officials have one allotted section:

$$2376 \text{ ft}^2$$

5. Diameter of the gold circle: $\frac{1}{3}(33) = 11$ inches

Radius of the gold circle: $\frac{11}{2} = 5.5$ inches

Diameter of the red circle: $\frac{2}{3}(33) = 22$ inches

Radius of the red circle: $\frac{22}{2} = 11$ inches

Step One:

$$\begin{aligned}&A_{\text{(total red)}}\pi r^2 - A_{\text{(gold)}}\pi r^2\\ &A_{\text{(total red)}}\pi(11)^2 - A_{\text{(gold)}}\pi(5.5)^2\\ &\approx 285.1 \text{ inches}^2\end{aligned}$$

Step Two:

$$\begin{aligned}&A_{\text{(red sector)}}\frac{1}{2}r^2\theta - A_{\text{(gold sector)}}\frac{1}{2}r^2\theta\\ &A_{\text{(red sector)}}\frac{1}{2}(11)^2\left(\frac{\pi}{4}\right) - A_{\text{(gold sector)}}\frac{1}{2}(5.5)^2\left(\frac{\pi}{4}\right)\\ &\approx 35.6 \text{ inches}^2\end{aligned}$$

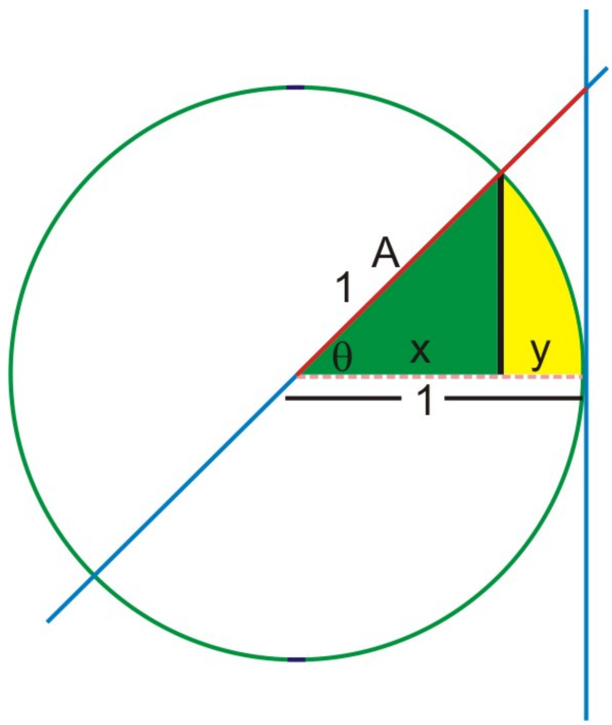
Step Three:

$$285.1 - 35.6 \approx 249.5 \text{ inches}^2$$

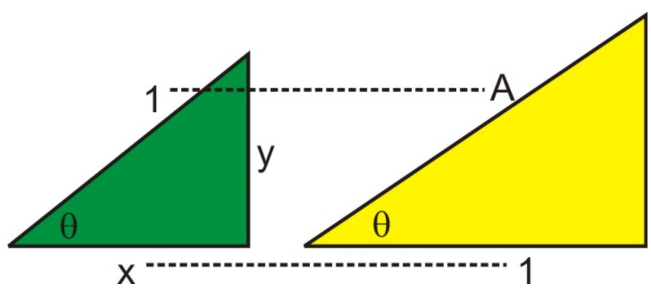
Circular Functions of Real Numbers

Review Exercises:

1.



Using similar triangles:

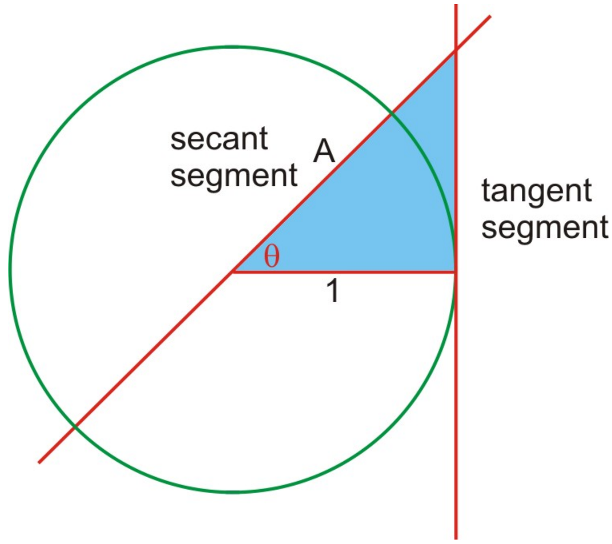


$$\frac{x}{1} = \frac{1}{A} \qquad A = \frac{1}{x}$$

$$Ax = 1 \qquad \cos\theta = x \qquad \frac{1}{\cos\theta} = \sec\theta$$

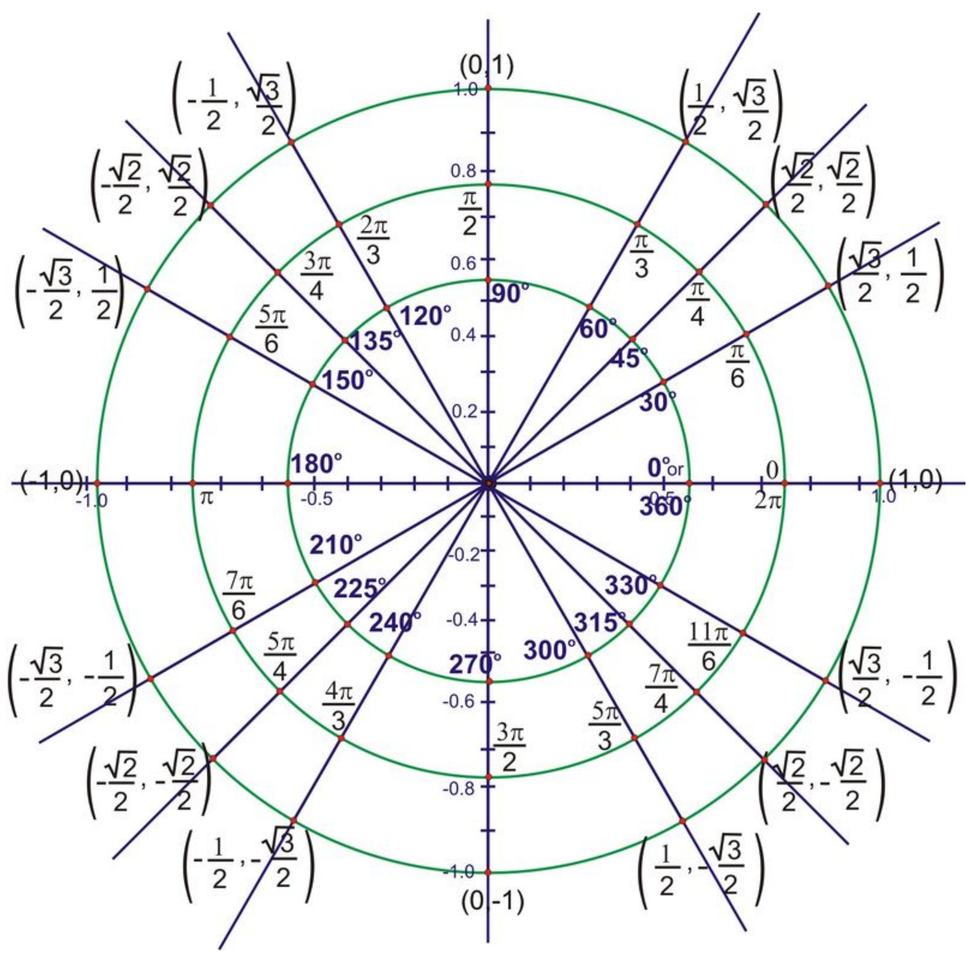
$$\frac{Ax}{x} = \frac{1}{x} \qquad \therefore A = \frac{1}{\cos\theta} \qquad \therefore A = \sec\theta$$

2.

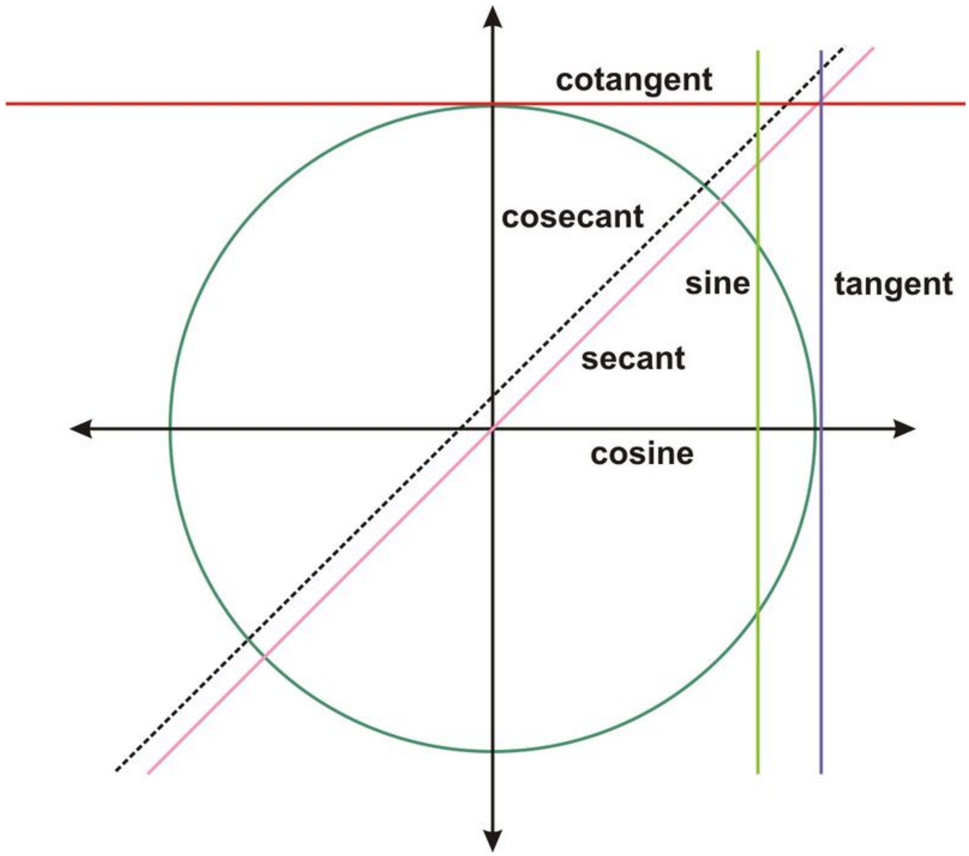


$$(h)^2 = (s_1)^2 + (s_2)^2$$
$$(\sec\theta)^2 = (1)^2 + (\tan\theta)^2$$
$$\sec^2\theta = 1 + \tan^2\theta$$

3.



4.



5. The $\tan(x)$ and $\sec(x)$ are two trigonometric functions that increase as x increases from 0 *to* $\frac{\pi}{2}$.

6. As x increases from $\frac{3\pi}{2}$ to 2π , $\cot(x)$ gets infinitely smaller.

Linear and Angular Velocity

Review Exercises

1. a)

$$c = 2\pi r \qquad v = \frac{s}{t}$$
$$c = 2\pi(7 \text{ cm.}) \qquad v = \frac{43.98}{9}$$
$$c \approx 43.98 \text{ cm} \qquad v \approx 4.89 \text{ cm/sec}$$

b) $w = \frac{\theta}{t}$ (where θ is one rotation (2π) and t is the time to complete 1 rotation)

$$w = \frac{2\pi}{9}$$
$$w \approx 0.698$$
$$w \approx 0.70 \text{ rad/sec}$$

2. a)

$$\begin{aligned}
v &= \frac{s}{t} \\
v &= \frac{43.98}{3.5} \\
v &\approx 12.57 \text{ cm/sec}
\end{aligned}$$

b) $w = \frac{\theta}{t}$ (where θ is one rotation (2π) and t is the time to complete 1 rotation)

$$\begin{aligned}
w &= \frac{2\pi}{3.5} \\
w &\approx 1.795 \\
w &\approx 1.80 \text{ rad/sec}
\end{aligned}$$

3. a) $w = \frac{\theta}{t}$ (where θ is one rotation (2π) and t is the time to complete 1 rotation)

$$\begin{aligned}
w &= \frac{2\pi}{12} \\
w &\approx 524 \\
w &\approx 0.524 \text{ rad/sec}
\end{aligned}$$

Velocity for Lois:	Velocity for Doris:
$v = rw$	$v = rw$
$v = (3m)(0.524)$	$v = (10m)(0.524)$
$v \approx 1.57$ m/sec	$v \approx 5.24$ m/sec

b) $w = \frac{\theta}{t}$ (where θ is one rotation (2π) and t is the time to complete 1 rotation)

$$\begin{aligned}
w &= \frac{2\pi}{12} \\
w &\approx 524 \\
w &\approx 0.524 \text{ rad/sec}
\end{aligned}$$

4. a)

$$\begin{aligned}
v &= \frac{s}{t} \\
t &= \frac{s}{v} \\
t &= \frac{2.7 \times 10^4}{3 \times 10^8} \\
t &\approx .9 \times 10^{-4} \approx 9.0 \times 10^{-5} \text{ seconds}
\end{aligned}$$

b) $w = \frac{\theta}{t}$ (where θ is one rotation (2π) and t is the time to complete 1 rotation)

$$w = \frac{2\pi}{9.0 \times 10^{-5}}$$
$$w \approx 69813.17 \text{ rad/sec} \approx 69813 \text{ rad/sec}$$

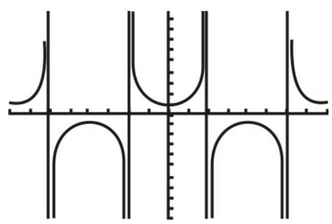
c)

$$\text{rotations} = \frac{v}{c} \text{ where v is the speed of the protons and c is the circumference of the LHC.}$$
$$\text{rotations} = \frac{3 \times 10^8}{2700}$$
$$\text{rotations} \approx 11,111 \text{ rotations in 1 second}$$

Graphing Sine and Cosine Functions

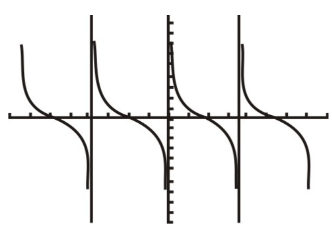
Review Exercises

1. The graph of $y = sec(x)$



The period is 2π and the frequency is 1 .

The graph of $y = \cot(x)$



The period is π and the frequency is 2 .

2.

TABLE 2.4:

Function	Minimum Value	Maximum Value
a) $y = \cos x$	-1	1
b) $y = 2\sin x$	-2	2
c) $y = -\sin x$	-1	1
d) $y = \tan x$	$-\infty$	$+\infty$

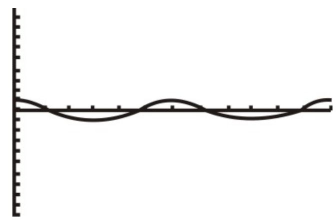
3. For the equation $4\sin(x) = \sin(x)$ over the interval $0 \leq x \leq 2\pi$ there are 3 real solutions.

4.

TABLE 2.5:

Function	Period	Amplitude	Frequency
$y = \cos(2x)$	π	1	2
$y = 3\sin x$	2π	3	1
$y = 2\sin(\pi x)$	$\frac{2\pi}{3}$	2	3
$y = 2\cos(3x)$	$\frac{2\pi}{3}$	2	3
$y = \frac{1}{2}\cos\left(\frac{1}{2}x\right)$	4π	$\frac{1}{2}$	$\frac{1}{2}$
$y = 3\sin\left(\frac{1}{2}x\right)$	4π	3	$\frac{1}{2}$

a) $y = \cos(2x)$

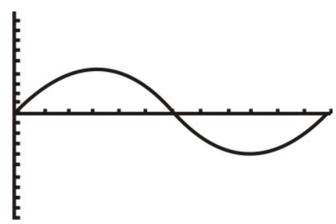


The period is π . This is the interval required to graph one complete cosine curve.

The amplitude is the distance from the sinusoidal axis to the maximum point of the curve. The amplitude of $y = \cos(2x)$ is 1 .

The frequency is the number of complete curves that are graphed over the interval of 2π . The frequency for $y = \cos(2x)$ is 2 .

b) $y = 3\ \sin x$

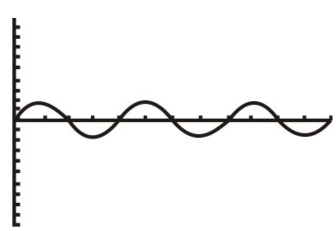


The period is 2π . This is the interval required to graph one complete sine curve.

The amplitude is the distance from the sinusoidal axis to the maximum point of the curve. The amplitude of $y = 3\ \sin(x)$ is 3 .

The frequency is the number of complete curves that are graphed over the interval of 2π . The frequency for $y = 3\sin(x)$ is 1 .

c) $y = 2\ \sin(\pi x)$

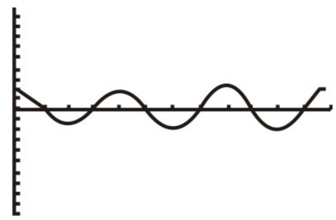


The period is $\frac{2\pi}{3}$. This is the interval required to graph one complete sine curve.

The amplitude is the distance from the sinusoidal axis to the maximum point of the curve. The amplitude of $y = 2\ \sin(\pi x)$ is 2 .

The frequency is the number of complete curves that are graphed over the interval of 2π . The frequency for $y = 2\ \sin(\pi x)$ is 3 .

d) $y = 2\ \cos(3x)$



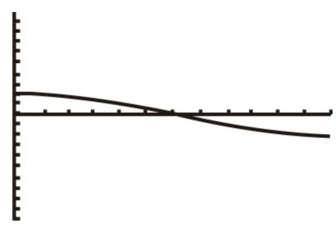
The period is $\frac{2\pi}{3}$. This is the interval required to graph one complete cosine curve

The amplitude is the distance from the sinusoidal axis to the maximum point of the curve. The amplitude of $y = 2\ \cos(3x)$ is 2 .

The frequency is the number of complete curves that are graphed over the interval of 2π . The frequency for $y = 2\ \cos(3x)$ is 3 .

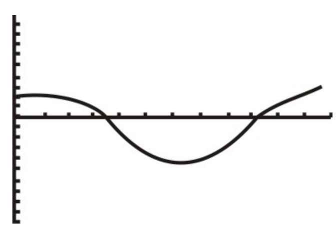
e) $y = \frac{1}{2}\cos\left(\frac{1}{2}x\right)$

Graph over 2π



The period is 4π . This is the interval required to graph one complete cosine curve

The amplitude is the distance from the sinusoidal axis to the maximum point of the curve. The amplitude of $y = \frac{1}{2}\cos\left(\frac{1}{2}x\right)$ is $\frac{1}{2}$.

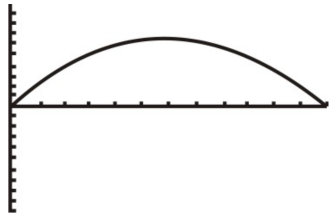


Graph over 4π

The frequency is the number of complete curves that are graphed over the interval of 2π . The frequency for $y = \frac{1}{2}\cos\left(\frac{1}{2}x\right)$ is $\frac{1}{2}$.

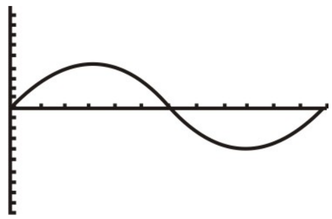
f) $y = 3\sin\left(\frac{1}{2}x\right)$

Graph over 2π



The period is 4π . This is the interval required to graph one complete sine curve.

The amplitude is the distance from the sinusoidal axis to the maximum point of the curve. The amplitude of $y = 3\ \sin\left(\frac{1}{2}x\right)$ is 3 .



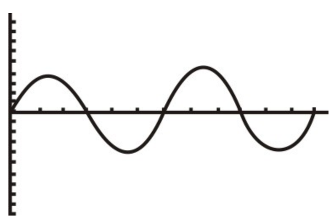
The frequency is the number of complete curves that are graphed over the interval of 2π . The frequency for $y = 3\ \sin\left(\frac{1}{2}x\right)$ is $\frac{1}{2}$.

5.

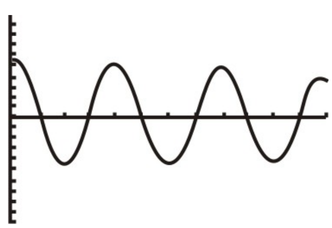
TABLE 2.6:

Period	Amplitude	Frequency	Equation
π	3	2	$y = 3\ \cos(2x)$
4π	2	$\frac{1}{2}$	$y = 2\ \sin\left(\frac{1}{2}x\right)$
$\frac{\pi}{2}$	2	4	$y = 2\ \cos(4x)$
$\frac{\pi}{3}$	$\frac{1}{2}$	6	$y = \frac{1}{2}\ \sin(6x)$

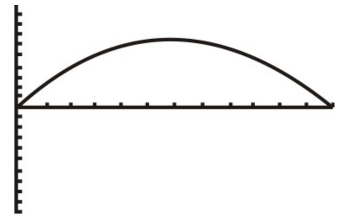
6. a) $y = 3\ \sin(2x)$



b) $y = 2.5\cos(\pi x)$



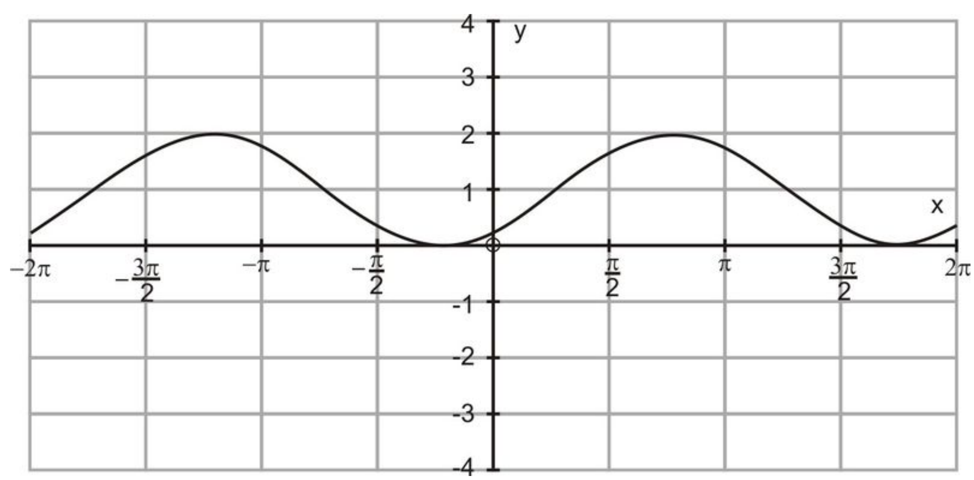
c) $y = 4\sin\left(\frac{1}{2}x\right)$



Translating Sine and Cosine Functions

Review Exercises

1. B the minimum value is 0. $A.\ y = \sin\left(x + \frac{\pi}{2}\right)$
2. E the maximum value is 3. $B.\ y = 1 + \sin(x)$
3. D the minimum value is -2. $C.\ y = \cos(x - \pi)$
4. C the y-intercept is -1. $D.\ y = -1 + \sin\left(x - \frac{3\pi}{2}\right)$
5. A the same graph as $y = cosx$ $E.\ y = 2 + \cos x$
6. $y = -2 + \sin(x + \pi)$ and $y = -2 + \cos\left(x + \frac{\pi}{2}\right)$
7. $y = 2 + \sin\left(x - \frac{\pi}{2}\right)$ Graph C
8. $y = -1 + \cos\left(x + \frac{3\pi}{2}\right)$ Graph D
9. $y = 2 + \cos\left(x - \frac{\pi}{2}\right)$ Graph A
10. $y = -1 + \sin(x - \pi)$ Graph B
11. The graph of $y = 1 + \sin\left(x - \frac{\pi}{4}\right)$



General Sinusoidal Functions

Review Exercises

The following general form of a sinusoidal function will be used to answer 1 – 5.

$y = C + A\sin(B(x - D))$ where: C represents the Vertical Translation(V.T.)

A represents the Vertical Stretch (amplitude) (V.S.)

B represents the Horizontal Stretch (H.S.)

D represents the Horizontal Translation (H.T.)

1. $y = 2 + 3\ \sin(2(x-1))$ The graph of this sinusoidal curve is the graph of $y = \sin x$ that has been vertically translated upward 2 units and horizontally translated I unit to the right. The amplitude of the curve is 3 and the period is $\frac{1}{2}(2\pi)$ or π . The frequency is 2 . The graph will have a maximum value of 5 and a minimum value of -1 .

2. $y = -1 + \sin(\pi(x + \frac{\pi}{3}))$ The graph of this sinusoidal curve is the graph of $y = \sin x$ that has been vertically translated downward 1 unit and horizontally translated $\frac{\pi}{3}$ units to the left. The amplitude of the curve is 1 and the period is 2 . The frequency is π . The graph will have a maximum value of 0 and a minimum value of -2 .

3. $y = \cos(40x - 120) + 5$ The graph of this sinusoidal curve is the graph of $y = \cos x$ that has been vertically translated upward 5 units and horizontally translated 30 radians to the right. The amplitude of the curve is 1 and the period is $\frac{\pi}{20}$. The frequency is 40 . The graph will have a maximum value of 5 and a minimum value of 4 .

4. $y = -\cos\left(\frac{1}{2}\left(x + \frac{5\pi}{4}\right)\right)$ The graph of this sinusoidal curve is the graph of $y = \cos x$ that has not been vertically translated but has been horizontally translated $\frac{5\pi}{4}$ radians to the left. The negative sign in front of the function indicates that the graph has been reflected across the $x-$ axis. The amplitude of the curve is 1 and the period is 4π . The frequency is $\frac{1}{2}$. The graph will have a maximum value of 1 and a minimum value of -1 .

5. $y = 3 + 2\ \cos(-x)$ The graph of this sinusoidal curve is the graph of $y = \cos x$ that has been vertically translated upward 3 units . There is no horizontal translation. However, the negative sign in front of the x indicates that the graph has been reflected across the $y-$ axis. This reflection is not visible in the graph since the graph is symmetric with the $y-$ axis. The amplitude of the curve is 2 and the period is 2π . The frequency is 1 . The graph will have a maximum value of 5 and a minimum value of 1 . All of the above answers can be confirmed by using the TI-83 to graph each function.

6. For this graph, the transformations of $y = \cos(x)$ are:

$$VR \to No; VS \to 2; VT \to 3$$
$$HS \to \frac{\pi}{2}\left(\frac{1}{2\pi}\right) = \frac{1}{4}; HT \to \frac{\pi}{6}$$

The equation that models the graph is $y = 3 + 2\ \cos\left(4\left(x - \frac{\pi}{6}\right)\right)$

7. For this graph, the transformations of $y = \sin(x)$ are:

$$VR \to No; VS \to 1; VT \to 2$$
$$HS \to \frac{2\pi}{2\pi} = 1; HT \to -\frac{3\pi}{2}$$

The equation that models the graph is $y = 2 + sin\left(x + \frac{3\pi}{2}\right)$

8. For this graph, the transformations of $y = \cos(x)$ are:

$$VR \to No; VS \to 20; VT \to 10$$
$$HS \to \frac{60^\circ}{360^\circ} = \frac{1}{6}; HT \to 30^\circ$$

The equation that models the graph is $y = 10 + 20\ \cos(6(x - 30^\circ))$

9. For this graph, the transformations of $y = \sin(x)$ are:

$$VR \to No; VS \to \frac{3}{4}; VT \to 3$$
$$HS \to \frac{4\pi}{2\pi} = 2; HT \to -\pi$$

The equation that models the graph is $y = 3 + \frac{3}{4}\sin\left(\frac{1}{2}(x+\pi)\right)$

10. For this graph, the transformations of $y = \cos(x)$ are:

$$VR \to No; VS \to 7; VT \to 3$$
$$HS \to \frac{12\pi}{4\pi} = 3; HT \to \frac{\pi}{4}$$

The equation that models the graph is $y = 3 + 7\cos\left(\frac{1}{3}\left(x - \frac{\pi}{4}\right)\right)$

CHAPTER 3 Trigonometric Identities - Solution Key

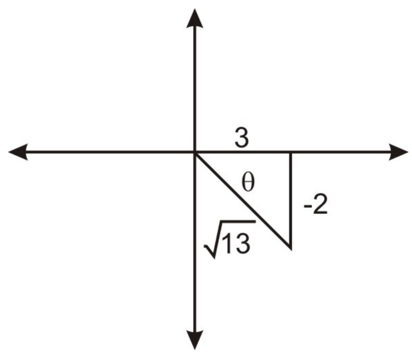
CHAPTER OUTLINE

3.1 Trigonometric Identities

Fundamental Identities

Review Exercises:

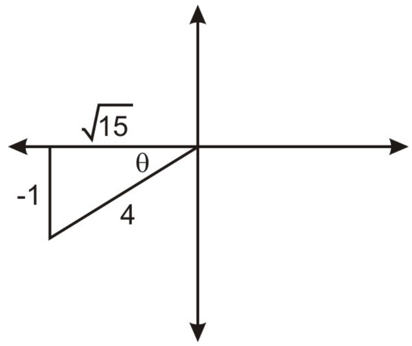
1. If the tangent of an angle has a negative value, then the angle must be found in either the 2nd or 4th quadrant. If $\cos\theta > 0$, then the angle must be located in the 4th quadrant since the cosine function is positive in this quadrant. Given $\theta = -\frac{2}{3}$ and this angle is located in the 4th quadrant, the negative value is 2 . To determine the value of $\sin\theta$, the length of the hypotenuse must be found by using Pythagorean Theorem.



$$
\begin{aligned}
(h)^2 &= (s_1)^2 + (s_2)^2 \\
(h)^2 &= (2)^2 + (-3)^2 \\
\sqrt{h^2} &= \sqrt{13} \\
h &= \sqrt{13}
\end{aligned}
$$

$$
\begin{aligned}
\sin\theta &= \frac{opp}{hyp} \\
\sin\theta &= \frac{-2}{\sqrt{13}} \rightarrow \sin\theta = \frac{-2}{\sqrt{13}}\left(\frac{\sqrt{13}}{\sqrt{13}}\right) \\
sin\theta &= -\frac{2\sqrt{13}}{13}
\end{aligned}
$$

2. If $\csc\theta = -4$ and $\sin\theta = \frac{1}{\csc\theta}$ then $\sin\theta = -\frac{1}{4}$. The sine function is negative in the 3rd and 4th quadrants. However, if $\tan\theta > 0$, then the angle must be in the 3rd quadrant since the value of the tangent function is positive in this quadrant. The length of the adjacent side must be found by using Pythagorean Theorem.

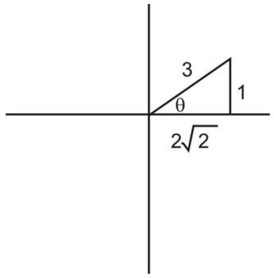


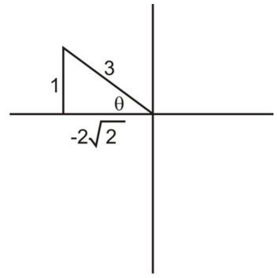
$$
\begin{aligned}
(h)^2 &= (s_1)^2 + (s_2)^2 \\
(4)^2 &= (-1)^2 + (s_2)^2 \\
16 &= 1 + (s_2)^2 \\
\sqrt{15} &= \sqrt{s^2} \\
\sqrt{15} &= s
\end{aligned}
$$

$$
\begin{array}{lll}
\sin\theta = \frac{opp}{hyp} & \cos\theta = \frac{adj}{hyp} & \tan\theta = \frac{opp}{adj} \\
\sin\theta = -\frac{1}{4} & \cos\theta = -\frac{\sqrt{15}}{4} & \tan\theta = \frac{1}{\sqrt{15}} \to \tan\theta = \frac{1}{\sqrt{15}}\left(\frac{\sqrt{15}}{\sqrt{15}}\right) \\
 & & \tan\theta = \frac{\sqrt{15}}{15} \\
\csc\theta = -4 & \sec\theta = -\frac{4}{\sqrt{15}} \to & \sec\theta = -\frac{4}{\sqrt{15}}\left(\frac{\sqrt{\sqrt{15}}}{\sqrt{15}}\right) \\
\sec\theta = -\frac{4\sqrt{15}}{15} & \cot\theta = \frac{\sqrt{15}}{1} &
\end{array}
$$

3. If $\sin\theta = \frac{1}{3}$, then the angles are located in the 1^{st} and 2^{nd} quadrants since the sine function is positive in these quadrants. There are also two values for the cosine function in these quadrants. The length of the adjacent side must be found by using Pythagorean Theorem.

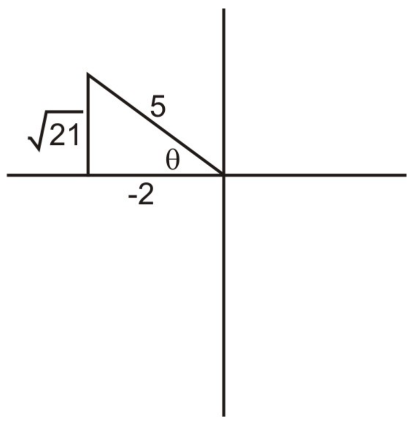
$$
\begin{aligned}
(h)^2 &= (s_1)^2 + (s_2)^2 \\
(3)^2 &= (1)^2 + (s_2)^2 \\
9 &= 1 + (s_2)^2 \\
\sqrt{8} &= \sqrt{s^2} \\
\sqrt{4 \cdot 2} &= s \\
2\sqrt{2} &= s
\end{aligned}
$$





$$\cos\theta = \frac{adj}{hyp} \qquad \cos\theta = \frac{adj}{hyp}$$
$$\cos\theta = \frac{2\sqrt{2}}{3} \qquad \cos\theta = -\frac{2\sqrt{2}}{3}$$

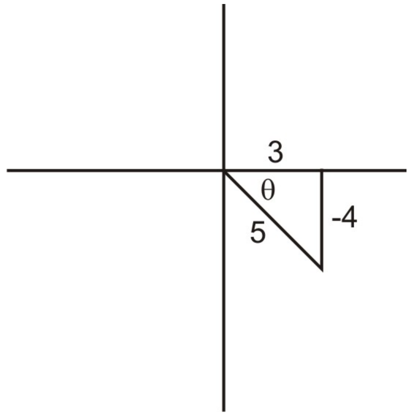
4. If $\cos\theta = -\frac{2}{5}$ and the angle is located in the 2^{nd} quadrant, the length of the opposite side must be determined in order to determine the values of the remaining trigonometric functions.



$$\begin{aligned}
(h)^2 &= (s_1)^2 + (s_2)^2 \\
(5)^2 &= (-2)^2 + (s_2)^2 \\
25 &= 4 + (s_2)^2 \\
\sqrt{21} &= \sqrt{s^2} \\
\sqrt{21} &= s
\end{aligned}$$

$$\sin\theta = \frac{opp}{hyp} \qquad \tan\theta = \frac{opp}{adj} \qquad \csc\theta = \frac{hyp}{opp}$$
$$\sin\theta = \frac{\sqrt{21}}{5} \qquad \tan\theta = -\frac{\sqrt{21}}{2} \qquad \csc\theta = \frac{5}{\sqrt{21}} = \frac{5}{\sqrt{21}}\left(\frac{\sqrt{21}}{\sqrt{21}}\right) = \frac{5\sqrt{21}}{21}$$
$$\sec\theta = \frac{hyp}{adj} \qquad \cot\theta = \frac{adj}{opp}$$
$$\sec\theta = -\frac{5}{2} \qquad \cot\theta = -\frac{2}{\sqrt{21}} = -\frac{2}{\sqrt{21}}\left(\frac{\sqrt{21}}{\sqrt{21}}\right) = -\frac{2\sqrt{21}}{21}$$

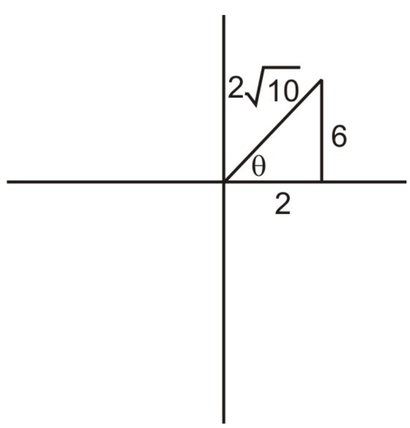
5. If $(3,-4)$ is on the terminal side of the angle in standard position, the angle is located in the 4^{th} quadrant. The Pythagorean Theorem can be used to determine the length of the hypotenuse.



$$
\begin{aligned}
(h)^2 &= (s_1)^2 + (s_2)^2 \\
(h)^2 &= (3)^2 + (-4)^2 \\
(h)^2 &= 9 + 16 \\
\sqrt{h^2} &= \sqrt{25} \\
h &= 5
\end{aligned}
$$

$$
\begin{array}{llllll}
\sin\theta = \frac{opp}{hyp} & \cos\theta = \frac{adj}{hyp} & \tan\theta = \frac{opp}{adj} & \csc\theta = \frac{hyp}{opp} & \sec\theta = \frac{hyp}{adj} & \cot\theta = \frac{adj}{opp} \\
\sin\theta = -\frac{4}{5} & \cos\theta = \frac{3}{5} & \tan\theta = -\frac{4}{3} & \csc\theta = -\frac{5}{4} & \sec\theta = \frac{5}{3} & \cot\theta = -\frac{3}{4}
\end{array}
$$

6. If $(2,6)$ is on the terminal side of the angle in standard position, the angle is located in the 1^{st} quadrant. The values of the trigonometric functions will all be positive. The Pythagorean Theorem can be used to determine the length of the hypotenuse.



$$
\begin{aligned}
(h)^2 &= (s_1)^2 + (s_2)^2 \\
(h)^2 &= (2)^2 + (6)^2 \\
(h)^2 &= 40 \\
\sqrt{h^2} &= \sqrt{40} \\
\sqrt{h^2} &= \sqrt{4 \cdot 10} \\
h &= 2\sqrt{10}
\end{aligned}
$$

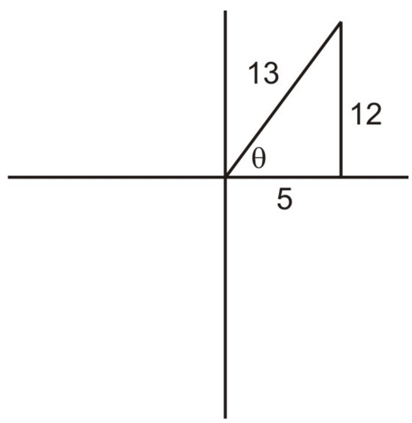
$$\sin\theta = \frac{opp}{hyp} \qquad \cos\theta = \frac{adj}{hyp}$$

$$\sin\theta = \frac{6}{2\sqrt{10}} = \frac{6}{2\sqrt{10}}\left(\frac{\sqrt{10}}{\sqrt{10}}\right) = \frac{3\sqrt{10}}{10} \qquad \cos\theta = \frac{2}{2\sqrt{10}} = \frac{2}{2\sqrt{10}}\left(\frac{\sqrt{10}}{\sqrt{10}}\right) = \frac{\sqrt{10}}{10}$$

$$\tan\theta = \frac{opp}{adj} \qquad \csc\theta = \frac{hyp}{opp} \qquad \sec\theta = \frac{hyp}{adj} \qquad \cot\theta = \frac{adj}{opp}$$

$$\tan\theta = \frac{6}{2} = 3 \qquad \csc\theta = \frac{2\sqrt{10}}{6} = \frac{\sqrt{10}}{3} \qquad \sec\theta = \frac{2\sqrt{10}}{2} = \sqrt{10} \qquad \cot\theta = \frac{2}{6} = \frac{1}{3}$$

7. a)

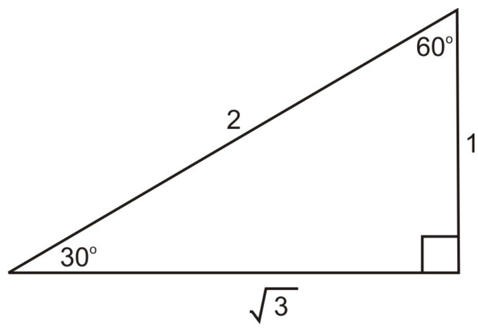


$$\sin\theta = \frac{opp}{hyp} \qquad \cos\theta = \frac{adj}{hyp}$$

$$\sin\theta = \frac{12}{13} \qquad \cos\theta = \frac{5}{13}$$

$$
\begin{aligned}
\sin^2\theta + \cos^2\theta &= 1 \\
\left(\frac{12}{13}\right)^2 + \left(\frac{5}{13}\right)^2 &= 1 \\
\frac{144}{169} + \frac{25}{169} &= 1 \\
\frac{169}{169} &= 1 \\
1 &= 1
\end{aligned}
$$

b.



$$\sin\theta = \frac{opp}{hyp} \qquad \cos\theta = \frac{adj}{hyp}$$
$$\sin\theta = \frac{1}{2} \qquad \cos\theta = \frac{\sqrt{3}}{2}$$

$$\begin{aligned}\sin^2\theta+\cos^2\theta &= 1\\ \left(\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2 &= 1\\ \frac{1}{4}+\frac{3}{4} &= 1\\ \frac{4}{4} &= 1\\ 1 &= 1\end{aligned}$$

8. a) To factor $\sin^2\theta-\cos^2\theta$, use the difference of squares. If this does not appear to be an obvious approach, let $x^2=\sin^2\theta$ and let $y^2=\cos^2\theta$ and factor x^2-y^2 .

$$\begin{aligned}&\sin^2\theta-\cos^2\theta && x^2-y^2\\ &(\sin\theta+\cos\theta)(\sin\theta-\cos\theta) && (x+y)(x-y)\to\sqrt{x^2}=\sqrt{\sin^2\theta}\to x=\sin\theta\\ &&&\to\sqrt{y^2}=\sqrt{\cos^2\theta}\to y=\cos\theta\\ &&&(\sin\theta+\cos\theta)(\sin\theta-\cos\theta)\end{aligned}$$

b)

$$\begin{aligned}&\sin^2\theta+6\sin\theta+8\\ &(\sin\theta+4)(\sin\theta+2)\end{aligned}$$

9. $\frac{\sin^4\theta-\cos^4\theta}{\sin^2\theta-\cos^2\theta}$ To simplify this expression, the first step is to factor the expression.

$$\begin{aligned}&\frac{(\sin^2\theta+\cos^2\theta)(\sin^2\theta-\cos^2\theta)}{(\sin^2\theta-\cos^2\theta)}\\ &\frac{(\sin^2\theta-\cos^2\theta)}{(\sin^2\theta-\cos^2\theta)}=1\to\text{substitute}(\sin^2\theta+\cos^2\theta=1)\end{aligned}$$

10. $\tan^2\theta+1=\sec^2\theta$ To prove this identity, use the quotient identity $\tan\theta=\frac{\sin\theta}{\cos\theta}$ and the reciprocal identity $\sec\theta=\frac{1}{\cos\theta}$

$$\begin{aligned}
\frac{\sin^2\theta}{\cos^2\theta}+1&=\frac{1}{\cos^2\theta} & \frac{\sin^2\theta+\cos^2\theta}{\cos^2\theta}&=\frac{1}{cos^2\theta}\\
\frac{\sin^2\theta}{\cos^2\theta}+1\left(\frac{\cos^2\theta}{\cos^2\theta}\right)&=\frac{1}{\cos^2\theta} & \frac{1}{\cos^2\theta}&=\frac{1}{\cos^2\theta}\rightarrow \text{substitute}(\sin^2\theta+\cos^2\theta=1)\\
\frac{\sin^2\theta}{\cos^2\theta}+\left(\frac{\cos^2\theta}{\cos^2\theta}\right)&=\frac{1}{\cos^2\theta}
\end{aligned}$$

Verifying Identities

Review Exercises:

To verify a trigonometric identity, it is often easier to work with only one side of the given equation. The goal will then be to have the left side read the same as the right side. Working with only one side of the equation means that the solution is always visible and there is no confusion as to what is being sought. This method will not work 100% of the time but it will work a lot of the time. If one side is kept constant, then all manipulations can be done to achieve the same constant on the working side.

1. Verify $\sin x\tan x+\cos x=\sec x$

$$\begin{aligned}
\sin x\tan x+\cos x&=\sec x\\
\sin x+\left(\frac{\sin x}{\cos x}\right)+\cos x&=\sec x\rightarrow\tan x=\frac{\sin x}{\cos x}\\
\frac{\sin^2 x}{\cos^2 x}+\cos x&=\sec x\\
\frac{\sin^2 x}{\cos^2 x}+\left(\frac{\cos x}{\cos x}\right)\cos x&=\sec x\rightarrow\text{common deno min ator}\\
\frac{\sin^2 x}{\cos x}+\frac{\cos^2 x}{\cos x}&=\sec x\\
\frac{\sin^2 x+\cos^2 x}{\cos x}&=\sec x\rightarrow\sin^2 x+\cos^2 x=1\\
\frac{1}{\cos x}&=\sec x\rightarrow\frac{1}{\cos x}=\sec x\\
\sec x&=\sec x
\end{aligned}$$

2. Verify $\cos x-\cos x\sin^2 x=\cos^3 x$

$$\begin{aligned}
\cos x-\cos x\sin^2 x&=\cos^3 x\rightarrow\text{remove the common factor }\cos x\\
\cos x(1-\sin^2 x)&=\cos^3 x\rightarrow\sin^2 x+\cos^2 x=1\rightarrow\cos^2 x=1-\sin^2 x\\
\cos x(\cos^2 x)&=\cos^3 x\\
\cos^3 x&=\cos^3 x
\end{aligned}$$

3. Verify $\frac{\sin x}{1+\cos x}+\frac{1+\cos x}{\sin x}=2\csc x$

$$\begin{aligned}
\frac{\sin x}{1+\cos x}+\frac{1+\cos x}{\sin x} &= 2\csc x \rightarrow \text{(common deno min ator)}\\
\left(\frac{\sin x}{\sin x}\right)\frac{\sin x}{1+\cos x}+\left(\frac{1+\cos x}{1+\cos x}\right)\frac{1+\cos x}{\sin x} &= 2\csc x \rightarrow \text{expand}\\
\frac{\sin^2 x}{(\sin x)(1+\cos x)}+\frac{1+2\cos x+\cos^2 x}{(\sin x)(1+\cos x)} &= 2\csc x \rightarrow \text{rearrange}\\
\frac{\sin^2 x+\cos^2 x+1+2\cos x}{(\sin x)(1+\cos x)} &= 2\csc x \rightarrow (\sin^2 x+\cos^2 x=1)\\
\frac{1+1+2\cos x}{(\sin x)(1+\cos x)} &= 2\csc x \rightarrow \text{simplify}\\
\frac{2+2\cos x}{(\sin x)(1+\cos x)} &= 2\csc x \rightarrow \text{(remove common factor)}\\
\frac{2(1+\cos x)}{(\sin x)(1+\cos x)} &= 2\csc x \rightarrow \text{simplify}\\
\frac{2}{(\sin x)} &= 2\csc x\\
2\frac{1}{(\sin x)} &= 2\csc x \rightarrow \csc x=\frac{1}{\sin x}\\
2\csc x &= 2\csc x
\end{aligned}$$

4. Verify $\frac{\sin x}{1+\cos x}=\frac{1-\cos x}{\sin x}$

$$\begin{aligned}
\frac{\sin x}{1+\cos x} &= \frac{1-\cos x}{\sin x} \rightarrow \text{cross multiply}\\
(\sin x)(\sin x) &= (1-\cos x)(1+\cos x)\\
sin^2 x &= 1-\cos^2 x \rightarrow \sin^2 x+\cos^2 x=1 \rightarrow \sin^2 x=1-\cos^2\\
\sin^2 x &= \sin^2 x
\end{aligned}$$

5. Verify $\frac{1}{1+\cos a}+\frac{1}{1-\cos a}=2+2\cot^2 a$

$$\begin{aligned}
\frac{1}{1+\cos a}+\frac{1}{1-\cos a} &= 2+2\cot^2 a \rightarrow \text{(common deno min ator)}\\
\left(\frac{1-\cos a}{1-\cos a}\right)\frac{1}{1+\cos a}+\left(\frac{1+\cos a}{1+\cos a}\right)\frac{1}{1-\cos a} &= 2+2\cot^2 a \rightarrow \text{multiply}\\
\frac{1-\cos a}{1-\cos^2 a}+\frac{1+\cos a}{1-\cos^2 a} &= 2+2cot^2 a \rightarrow \text{simplify}\\
\frac{1-\cos a+1+cosa}{1-\cos^2 a} &= 2+2cot^2 a \rightarrow \text{simplify} \rightarrow \sin^2 a+\cos^2 a=1\\
&\rightarrow \sin^2 a = 1-\cos^2 a\\
\frac{2}{\sin^2} &= 2+2\cot^2 a \rightarrow \text{(remove common factor)}\\
\frac{2}{\sin^2} &= 2(1+\cot^2 a) \rightarrow \cot a = \frac{\cos a}{\sin a}\\
\frac{2}{\sin^2} &= 2\left(1+\frac{cos^2 a}{\sin^2}\right) \rightarrow \text{(common deno min ator)}\\
\frac{2}{\sin^2} &= 2\left(\left(\frac{\sin^2 a}{\sin^2 a}\right)1+\frac{\cos^2 a}{\sin^2 a}\right) \rightarrow \text{multiply}\\
\frac{2}{\sin^2} &= 2\left(\frac{\sin^2 a}{\sin^2 a}+\frac{\cos^2 a}{\sin^2 a}\right) \rightarrow \text{simplify}\\
\frac{2}{\sin^2} &= 2\left(\frac{\sin^2 a+\cos^2 a}{\sin^2 a}\right) \rightarrow \sin^2 a+\cos^2 a=1\\
\frac{2}{\sin^2} &= 2\left(\frac{1}{\sin^2 a}\right)\\
\frac{2}{\sin^2} &= \frac{2}{\sin^2 a}
\end{aligned}$$

6. Verify $\cos^4 b-\sin^4 b = 1-2\sin^2 b$

$$\begin{aligned}
\cos^4 b-\sin^4 b &= 1-2\sin^2 b \rightarrow \text{factor}\\
(\cos^2 b-\sin^2 b)(\cos^2 b+\sin^2 b) &= 1-2\sin^2 b \rightarrow \sin^2 a+\cos^2 a=1\\
(\cos^2 b-\sin^2 b) &= 1-2\sin^2 b \rightarrow \sin^2 b+\cos^2 b=1 \rightarrow \cos^2 b=1-\sin^2 b\\
(1-\sin^2 b-\sin^2 b) &= 1-2\sin^2 b \rightarrow \text{simplify}\\
1-2\sin^2 b &= 1-2\sin^2 b
\end{aligned}$$

7. Verify $\frac{\sin y+\cos y}{\sin y}-\frac{\cos y-siny}{cosy} = \sec y\csc y$

$$\begin{aligned}
\frac{\sin y+\cos y}{\sin y}-\frac{\cos y-\sin y}{\cos y} &= \sec y\csc y \to \text{common denominator}\\
\left(\frac{\cos y}{cosy}\right)\frac{\sin y+\cos y}{\sin y}-\left(\frac{\sin y}{\sin y}\right)\frac{\cos y-\sin y}{\cos y} &= \sec y\csc y \to \text{nultiply}\\
\frac{\cos y\sin y+\cos^2 y}{\cos y\sin y}-\frac{\cos y\sin y+\sin^2 y}{\cos y\sin y} &= \sec y\csc y \to \text{rearrange}\\
\frac{\cos y\sin y-\cos y\sin y+\sin^2 y+\cos^2 y}{\cos y\sin y} &= \sec y\csc y \to \text{simplify} \to \sin^2 y+\cos^2 y=1\\
\frac{1}{\cos y\sin y} &= \sec y\csc y \to \text{express}\frac{1}{\cos y\sin y}\text{ as factors}\\
\frac{1}{\cos y}\cdot\frac{1}{\sin y} &= \sec\csc y \to \frac{1}{\cos y}=\sec y \text{ and } \frac{1}{\sin y}=\csc y\\
\sec y\csc y &= \sec y\csc y
\end{aligned}$$

8. Verify $(\sec x-\tan x^2)^2=\frac{1-\sin x}{1+\sin x}$

$$\begin{aligned}
(\sec x-\tan x)^2 &= \frac{1-\sec x}{1+\sin x} \to \text{expand}\\
\sec^2 x-2\sec x\tan x+\tan^2 &= \frac{1-\sec x}{1+\sin x} \to \sec x=\frac{1}{\cos x} \text{ and } \tan x=\frac{\sin x}{\cos x}\\
\frac{1}{\cos^2 x}-2\left(\frac{1}{\cos x}\right)\left(\frac{\sin x}{\cos x}\right)+\frac{\sin^2 x}{\cos^2 x} &= \frac{1-\sec x}{1+\sin x} \to \text{simplify}\\
\frac{1}{\cos^2 x}-2\frac{\sin x}{\cos^2 x}+\frac{\sin^2 x}{\cos^2 x} &= \frac{1-\sin x}{1+\sin x} \to \text{simplify}\\
\frac{1-2\sin x+\sin^2 x}{\cos^2 x} &= \frac{1-\sin x}{1+\sin x} \to \text{factor}\\
&\quad \to \sin^2 x+\cos^2 x=1-\sin^2 x\\
\frac{(1-\sin x)(1-\sin x)}{1-\sin^2 x} &= \frac{1-\sin x}{1+\sin x} \to \text{factor}\\
\frac{(1-\sin x)(1-\sin x)}{(1-\sin x)(1+\sin x)} &= \frac{1-\sin x}{1+\sin x} \to \text{simplify}\\
\frac{1-\sin x}{1+\sin x} &= \frac{1-\sin x}{1+\sin x}
\end{aligned}$$

9. To show that $2\sin x\cos x=\sin 2x$ for $\frac{5\pi}{6}$ substitute this value in for x and then use the values from the unit circle to simplify the expression.

$$
\begin{aligned}
2\sin x\cos x &= \sin 2x \\
2\sin\frac{5\pi}{6}\cos\frac{5\pi}{6} &= \sin 2\left(\frac{5\pi}{6}\right) \rightarrow \frac{5\pi}{6} = \frac{1}{2}; \cos\frac{5\pi}{6} \qquad = -\frac{\sqrt{3}}{2}; 2\left(\frac{5\pi}{6}\right) = \frac{5\pi}{3} \\
2\left(\frac{1}{2}\right)\cdot\left(-\frac{\sqrt{3}}{2}\right) &= \sin\frac{5\pi}{6} \rightarrow \sin\frac{5\pi}{3} = -\frac{\sqrt{3}}{2} \\
2\left(-\frac{\sqrt{3}}{4}\right) &= -\frac{\sqrt{3}}{2} \rightarrow \text{simplify} \\
-\frac{\sqrt{3}}{2} &= -\frac{\sqrt{3}}{2}
\end{aligned}
$$

10. Verify $\sec x \cot x = \csc x$

$$
\begin{aligned}
\sec x\cot x = \csc x \rightarrow \sec x &= \frac{1}{\cos x}; \cot x = \frac{\cos x}{\sin x} \\
\frac{1}{\cos x}\left(\frac{\cos x}{\sin x}\right) &= \csc x \rightarrow \text{simplify} \\
\frac{1}{\sin x} &= \csc x \rightarrow \csc x = \frac{1}{\sin x} \\
\csc x &= \csc x
\end{aligned}
$$

Sum and Difference Identities for Cosine

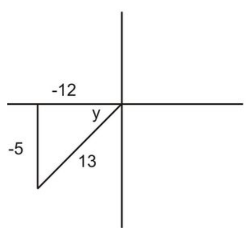
Review Exercises: Pages 246 – 250

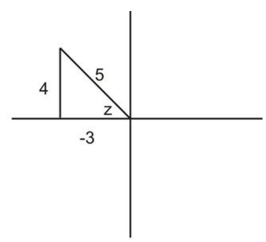
1. To calculate the exact value of $\cos\frac{5\pi}{12}$, the angle must be expressed in the form of the sum of two special angles. Once this is done, then the exact value can be determined by using the values for these angles. The unit circle can be used to determine these values. (Hint: It may be easier to convert the measure of the angle to degrees)

$\frac{5\pi}{12} = \frac{5(180)^\circ}{12} = 75^\circ$ Two special angles that add to 750 are 450 and 300 . These values can now be converted back to radians or the degrees may be used. $45^\circ\left(\frac{\pi}{180^\circ}\right) = \frac{\pi}{4}$ and $30\left(\frac{\pi}{180^\circ}\right) = \frac{\pi}{6}$

$$
\begin{aligned}
\frac{5\pi}{12} &= \left(\frac{\pi}{4}+\frac{\pi}{6}\right)\\
\cos(a+b) &= \cos a\cos b-\sin a\sin b\\
\cos\left(\frac{\pi}{4}+\frac{\pi}{6}\right) &= \left(\cos\frac{\pi}{4}\right)\left(\cos\frac{\pi}{6}\right)-\left(\sin\frac{\pi}{4}\right)\left(\sin\frac{\pi}{6}\right)\\
\cos\left(\frac{\pi}{4}+\frac{\pi}{6}\right) &= \left(\frac{1}{\sqrt{2}}\right)\left(\frac{\sqrt{3}}{2}\right)-\left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{2}\right)\\
\cos\left(\frac{\pi}{4}+\frac{\pi}{6}\right) &= \frac{\sqrt{3}}{2\sqrt{2}}-\frac{1}{2\sqrt{2}}\\
\cos\left(\frac{\pi}{4}+\frac{\pi}{6}\right) &= \frac{\sqrt{3}}{2\sqrt{2}}-\frac{1}{2\sqrt{2}}\\
\cos\left(\frac{\pi}{4}+\frac{\pi}{6}\right) &= \frac{\sqrt{3-1}}{2\sqrt{2}}\to\text{rationalize denominator}\\
\cos\left(\frac{\pi}{4}+\frac{\pi}{6}\right) &= \left(\frac{\sqrt{2}}{\sqrt{2}}\right)\left(\frac{\sqrt{3-1}}{2\sqrt{2}}\right)\\
\cos\left(\frac{\pi}{4}+\frac{\pi}{6}\right) &= \frac{\sqrt{6}-\sqrt{2}}{2\sqrt{4}}=\frac{\sqrt{6}-\sqrt{2}}{4}\to\text{simplify}
\end{aligned}
$$

2. To begin this question, sketch the two angles in standard position and use the Pythagorean Theorem to calculate the length of the adjacent side.



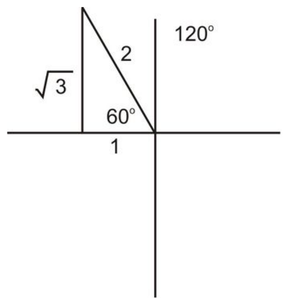


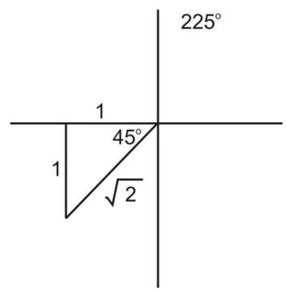
$$
\begin{aligned}
(h)^2 &= (s_1)^2+(s_2)^2\\
(13)^2 &= (12)^2+(s_2)^2\\
169 &= 144+(s_2)^2\\
\sqrt{25} &= \sqrt{s^2}\\
5 &= s
\end{aligned}
\qquad
\begin{aligned}
(h)^2 &= (s_1)^2+(s_2)^2\\
(5)^2 &= (3)^2+(s_2)^2\\
25 &= 9+(s_2)^2\\
\sqrt{16} &= \sqrt{s^2}\\
4 &= s
\end{aligned}
$$

Now, the value of $\cos(y-z)$ can be determined.

$$
\begin{aligned}
\cos(y-z) &= \cos y\cos z+\sin y\sin z\\
\cos(y-z) &= \left(-\frac{5}{13}\right)\left(\frac{4}{5}\right)+\left(\frac{12}{13}\right)\left(\frac{3}{5}\right)\\
\cos(y-z) &= -\frac{20}{65}+\frac{36}{65}\\
\cos(y-z) &= \frac{16}{65}
\end{aligned}
$$

3. There is more than one combination that could be used to determine the exact value of 345° . Two possible combinations are: $345^\circ = (300^\circ + 45^\circ) 345^\circ = (120^\circ + 225^\circ)$. Both of these will result in the same solution.



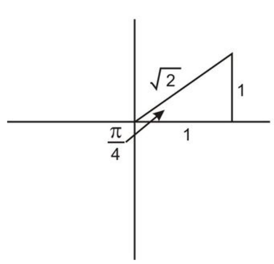


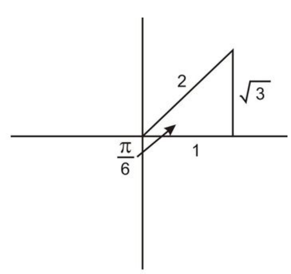
$\cos(a+b) = \cos a \cos b - \sin a \sin b$

$$\cos(120^\circ + 225^\circ) = \cos 120^\circ \cos 225^\circ - \sin 120^\circ \sin 225^\circ$$
$$\cos(120^\circ + 225^\circ) = \left(-\frac{1}{2}\right)\left(-\frac{1}{\sqrt{2}}\right) - \left(\frac{\sqrt{3}}{2}\right)\left(-\frac{1}{\sqrt{2}}\right)$$
$$\cos(120^\circ + 225^\circ) = \frac{1}{2\sqrt{2}} + \frac{\sqrt{3}}{2\sqrt{2}}$$
$$\cos(120^\circ + 225^\circ) = \frac{1+\sqrt{3}}{2\sqrt{2}} \rightarrow \text{rationalize denominator}$$
$$\cos(120^\circ + 225^\circ) = \left(\frac{\sqrt{2}}{\sqrt{2}}\right)\left(\frac{1+\sqrt{3}}{2\sqrt{2}}\right)$$
$$\cos(120^\circ + 225^\circ) = \frac{\sqrt{2}+\sqrt{6}}{2\sqrt{4}} \rightarrow \text{simplify}$$
$$\cos(120^\circ + 225^\circ) = \frac{\sqrt{2}+\sqrt{6}}{2\sqrt{4}}$$

4. $\cos 80 \cos 20 + \sin 80 \sin 20$ is the result of $\cos(a-b) = \cos a \cos b + \sin a \sin b$. Therefore the angle is $\cos(80 - 20) = \cos 60$. The value of $\cos 60$ is $\frac{1}{2}$.

5. The exact value of $\cos \frac{7\pi}{12}$ determined by calculating the sum of $\frac{3\pi}{12}$ and $\frac{4\pi}{12}$. These are two of the special angles. The angle $\frac{3\pi}{12} = \frac{\pi}{4}$ and the angle $\frac{4\pi}{12} = \frac{\pi}{3}$. Therefore, the exact value can be determined by using the cosine identity for the sum of two angles: $\cos(a+b) = \cos a \cos b - \sin a \sin b$





$$\cos(a+b) = \cos a \cos b - \sin a \sin b$$
$$\cos\left(\frac{\pi}{4}+\frac{\pi}{3}\right) = \left(\cos\frac{\pi}{4}\right)\left(\cos\frac{\pi}{3}\right) - \left(\sin\frac{\pi}{4}\right)\left(\sin\frac{\pi}{3}\right)$$
$$\cos\left(\frac{\pi}{4}+\frac{\pi}{3}\right) = \left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{2}\right) - \left(\frac{1}{\sqrt{2}}\right)\left(\frac{\sqrt{3}}{2}\right) \to \text{multiply}$$
$$\cos\left(\frac{\pi}{4}+\frac{\pi}{3}\right) = \frac{1}{2\sqrt{2}} - \frac{\sqrt{3}}{2\sqrt{2}} \to \text{rationalize deniminator}$$
$$\cos\left(\frac{\pi}{4}+\frac{\pi}{3}\right) = \left(\frac{\sqrt{2}}{\sqrt{2}}\right)\left(\frac{1}{2\sqrt{2}}\right) - \left(\frac{\sqrt{2}}{\sqrt{2}}\right)\left(\frac{\sqrt{3}}{2\sqrt{2}}\right) \to \text{simplify}$$
$$\cos\left(\frac{\pi}{4}+\frac{\pi}{3}\right) = \frac{\sqrt{2}}{2\sqrt{4}} - \frac{\sqrt{6}}{2\sqrt{4}}$$
$$\cos\left(\frac{\pi}{4}+\frac{\pi}{3}\right) = \frac{\sqrt{2}-\sqrt{6}}{4}$$

6. Verify $\frac{\cos(m-n)}{\sin m \cos n} = \cot m + \tan n$

To verify this identity $\cos(m-n)$ must be expanded using the cosine identity for the difference of two angles. In addition, cot m and tan n must be expressed in terms of sine and cosine. The next step will be to work with the right side of the equation so that it reads the same as the left side.

$$\frac{\cos(m-n)}{\sin m \cos n} = \cot m + \tan n$$
$$\frac{\cos m \cos n + \sin m \sin n}{\sin m \cos n} = \frac{\cos m}{\sin m} + \frac{\sin n}{\cos n} \to \text{common denominator (RS)}$$
$$\frac{\cos m \cos n + \sin m \sin n}{\sin m \cos n} = \left(\frac{\cos n}{\cos n}\right)\left(\frac{\cos m}{\sin m}\right) + \left(\frac{\sin m}{\sin m}\right)\left(\frac{\sin n}{\cos n}\right) \to \text{multiply}$$
$$\frac{\cos m \cos n + \sin m \sin n}{\sin m \cos n} = \frac{\cos m \cos n}{\sin m \cos n} + \frac{\sin m \sin n}{\sin m \cos n} \to \text{simplify}$$
$$\frac{\cos m \cos n + \sin m \sin n}{\sin m \cos n} = \frac{\cos m \cos n + \sin m \sin n}{\sin m \cos n}$$

7. Prove $\cos(\pi+\theta) = -\cos\theta$

To prove the above expression is simply a matter of using the cosine identity for the sum of two angles.

$$\cos(a+b) = \cos a \cos b - \sin a \sin b$$
$$\cos(\pi+\theta) = \cos\pi\cos\theta - \sin\pi\sin\theta$$
$$\cos(\pi+\theta) = (-1)\cos\theta - (0)\sin\theta$$
$$\cos(\pi+\theta) = -\cos\theta$$

8. Verify $\frac{\cos(c+d)}{\cos(c-d)} = \frac{1-\tan c \tan d}{1+\tan c \tan d}$

To verify this identity, the cosine identity for both the sum and the difference of angles must be used. As well, the quotient identity for tangent must be applied.

$$\begin{aligned}
\frac{\cos(c+d)}{\cos(c-d)} &= \frac{1-\tan c\tan d}{1+\tan c\tan d} \\
\frac{\cos c\cos d-\sin c\sin d}{\cos c\cos d+\sin c\sin d} &= \frac{1-\tan c\tan d}{1+\tan c\tan d} \rightarrow \div(\text{LS}) \text{ by} \cos c\cos d \\
\frac{\frac{\cos c\cos d-\sin c\sin d}{\cos c\cos d+\cos c\cos d}}{\frac{\cos c\cos d-\sin c\sin d}{\cos c\cos d+\cos c\cos d}} &= \frac{1-\tan c\tan d}{1+\tan c\tan d} \rightarrow \frac{\sin c}{\cos c} = \tan c\frac{\sin d}{\cos d} = \tan d \\
\frac{1-\tan c\tan d}{1+\tan c\tan d} &= \frac{1-\tan c\tan d}{1+\tan c\tan d}
\end{aligned}$$

9. To show that $\cos(a+b)\cdot\cos(a-b) = \cos^2 a - \sin^2 b$, the cosine identity for both the sum and the difference of angles must be used. Then the Pythagorean identity $\sin^2\theta + \cos^2\theta = 1$, or a form of this identity, will be applied.

$$\begin{aligned}
\cos(a+b)\cdot\cos(a-b) &= \cos^2 a - \sin^2 b \\
\cos(a+b)\cdot\cos(a-b) &= \cos^2 a - \sin^2 b \\
(\cos a\cos b-\sin a\sin b)(\cos a\cos b+\sin a\sin b) &= \cos^2 a - \sin^2 b \rightarrow \text{multiply} \\
\cos^2 a\cos^2 b-\sin a\sin b\cos a\cos b+\sin a\sin b\cos a\cos b-\sin^2 a\sin^2 b &\rightarrow \text{simplify} \\
\cos^2 a\cos^2 b-\sin^2 a\sin^2 b &= \cos^2 a - \sin^2 b \rightarrow \cos^2 b = 1-\sin^2 b \\
&\rightarrow \sin^2 a = 1-\cos^2 a \\
\cos^2 a(1-\sin^2 b)-(1-\cos^2 a)\sin^2 b &= \cos^2 a - \sin^2 b \rightarrow \text{expand} \\
\cos^2 a-\cos^2 a\sin^2 b-\sin^2 b+\cos^2 a\sin^2 b &= \cos^2 a - \sin^2 b \rightarrow \text{simplify} \\
\cos^2 a - \sin^2 b &= \cos^2 a - \sin^2 b
\end{aligned}$$

10. To determine all the solutions to the trigonometric equation $2\cos^2\left(x+\frac{\pi}{2}\right) = 1$ such that $0 \le x \le 2\pi$, it is necessary to first calculate the value of $\cos\left(x+\frac{\pi}{2}\right)$ and then to apply the cosine identity for the sum of angles.

$$\begin{aligned}
2\cos^2\left(x+\frac{\pi}{2}\right) &= 1 \\
2\cos^2\left(x+\frac{\pi}{2}\right) &= 1 \rightarrow \div \text{ both sides by 2} \\
\cos^2\left(x+\frac{\pi}{2}\right) &= \frac{1}{2} \rightarrow \sqrt{\text{both sides}} \\
\sqrt{\cos^2\left(x+\frac{x}{2}\right)} &= \sqrt{\frac{1}{2}} \rightarrow \text{rationalize denominator} \\
\cos\left(x+\frac{x}{2}\right) &= \left(\frac{\sqrt{2}}{\sqrt{2}}\right)\left(\sqrt{\frac{1}{2}}\right) \\
\cos\left(x+\frac{x}{2}\right) &= \frac{\sqrt{2}}{\sqrt{4}} \rightarrow \text{simplify} \\
\cos\left(x+\frac{x}{2}\right) &= \frac{\sqrt{2}}{2}
\end{aligned}$$

Now apply $\cos(a+b) = \cos a\cos b - \sin a\sin b$

$$\cos(a+b) = \cos a \cos b - \sin a \sin b$$
$$\cos\left(x+\frac{\pi}{2}\right) = \cos x \cos\frac{\pi}{2} - \sin x \sin\frac{\pi}{2}$$
$$\cos\left(x+\frac{\pi}{2}\right) = \cos x(0) - \sin x(1)$$
$$\cos\left(x+\frac{\pi}{2}\right) = -\sin x$$
$$\sin x = \frac{\sqrt{2}}{2} \rightarrow \div \text{ by } -1$$
$$\sin x = -\frac{\sqrt{2}}{2}$$

The sine function is negative in the 3rd and 4th quadrants. Therefore, there are 2 angles that have the value of sine equal to $-\frac{\sqrt{2}}{2}$. These angles are $\frac{5\pi}{4}$ and $\frac{7\pi}{4}$.

Sum and Difference Identities for Sine and Tangent

Review Exercises:

1. To find the exact value of $\frac{17\pi}{12}$, there is more than one combination that can be used. The solution presented here will use:

$$\sin\frac{17\pi}{12} = \sin\left(\frac{14\pi}{12} + \frac{3\pi}{12}\right)$$
$$\sin\frac{17\pi}{12} = \sin\left(\frac{7\pi}{6} + \frac{\pi}{4}\right)$$

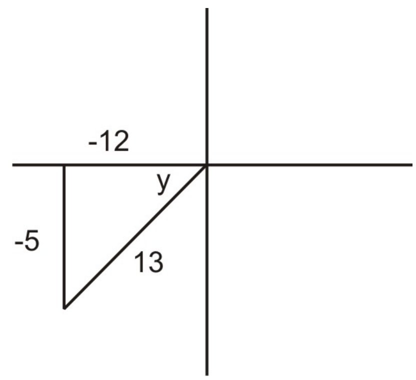
and the sine identity for the sum of angles $\sin(a+b) = \sin a \cos b + \cos a \sin b$, will be applied to determine the exact value.

$$\sin(a+b) = \sin a \cos b + \cos a \sin b$$
$$\sin\left(\frac{7\pi}{6}+\frac{\pi}{4}\right) = \left(\sin\frac{7\pi}{6}\right)\left(\cos\frac{\pi}{4}\right)+\left(\cos\frac{7\pi}{6}\right)\left(\sin\frac{\pi}{4}\right)$$
$$\sin\left(\frac{7\pi}{6}+\frac{\pi}{4}\right) = \left(-\frac{1}{2}\right)\left(\frac{1}{\sqrt{2}}\right)+\left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{\sqrt{2}}\right) \rightarrow \text{multiply}$$
$$\sin\left(\frac{7\pi}{6}+\frac{\pi}{4}\right) = -\frac{1}{2\sqrt{2}}-\frac{\sqrt{3}}{2\sqrt{2}} \rightarrow \text{simplify}$$
$$\sin\left(\frac{7\pi}{6}+\frac{\pi}{4}\right) = \frac{-1-\sqrt{3}}{2\sqrt{2}} \rightarrow \text{rationalize denominator}$$
$$\sin\left(\frac{7\pi}{6}+\frac{\pi}{4}\right) = \left(\frac{\sqrt{2}}{\sqrt{2}}\right)\left(\frac{-1-\sqrt{3}}{2\sqrt{2}}\right)$$
$$\sin\left(\frac{7\pi}{6}+\frac{\pi}{4}\right) = \frac{-\sqrt{2}-\sqrt{6}}{2\sqrt{4}} \rightarrow \text{simplify}$$
$$\sin\left(\frac{7\pi}{6}+\frac{\pi}{4}\right) = \frac{-\sqrt{2}-\sqrt{6}}{2\sqrt{4}}$$

2. To determine the exact value of sin 345° , the sine identity for the sum of angles can be used to find the sum of $(300^\circ+45^\circ)$.

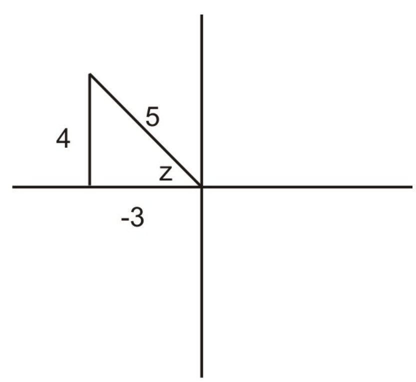
$$\sin(300^\circ+45^\circ) = \sin 300^\circ \cos 45^\circ - \cos 300^\circ \sin 45^\circ$$
$$\sin(300^\circ+45^\circ) = \left(-\frac{\sqrt{3}}{2}\right)\left(\frac{1}{\sqrt{2}}\right)+\left(\frac{1}{2}\right)\left(-\frac{1}{\sqrt{2}}\right) \rightarrow \text{multiply}$$
$$\sin(300^\circ+45^\circ) = \frac{\sqrt{3}}{2\sqrt{2}}+\frac{1}{2\sqrt{2}} \rightarrow \text{simplify}$$
$$\sin(300^\circ+45^\circ) = \frac{-\sqrt{3}+1}{2\sqrt{2}} \rightarrow \text{rationalize denominator}$$
$$\sin(300^\circ+45^\circ) = \left(\frac{\sqrt{2}}{\sqrt{2}}\right)\left(\frac{-\sqrt{3}+1}{2\sqrt{2}}\right)$$
$$\sin(300^\circ+45^\circ) = \frac{-\sqrt{6}+\sqrt{2}}{2\sqrt{4}} \rightarrow \text{simplify}$$
$$\sin(300^\circ+45^\circ) = \frac{-\sqrt{6}+\sqrt{2}}{4}$$

3. If $\sin y = -\frac{5}{13}$ and is located in the 3rd quadrant and $\sin z = \frac{4}{5}$ and is located in the 2nd quadrant, the value of the adjacent side can be found by using the Pythagorean Theorem. Then the value of $\sin(y+z)$ can be determined by using the sine identity for the sum of angles.



$$
\begin{aligned}
(h)^2 &= (s_1)^2 + (s_2)^2 \\
(13)^2 &= (-5)^2 + (s_2)^2 \\
169 &= 25 + (s_2)^2 \\
\sqrt{144} &= \sqrt{s^2} \\
12 &= s
\end{aligned}
$$

In the 3rd quadrant this value is negative.



$$
\begin{aligned}
(h)^2 &= (s_1)^2 + (s_2)^2 \\
(5)^2 &= (4)^2 + (s_2)^2 \\
25 &= 16 + (s_2)^2 \\
\sqrt{9} &= \sqrt{s^2} \\
3 &= s
\end{aligned}
$$

In the 2nd quadrant this value is negative.

$$
\begin{aligned}
\sin(y+z) &= \sin y \cos z + \cos y \sin z \\
\sin(y+z) &= \left(-\frac{5}{13}\right)\left(-\frac{3}{5}\right) + \left(-\frac{12}{13}\right)\left(\frac{4}{5}\right) \rightarrow \text{multiply} \\
\sin(y+z) &= \frac{15}{65} - \frac{48}{65} \rightarrow \text{simplify} \\
\sin(y+z) &= -\frac{33}{65}
\end{aligned}
$$

4. $\sin 25^o \cos 5^o + \cos 25^o \sin 5^o$ is the expanded form of $\sin(a+b)$. Therefore the angle is $\sin(25^o + 5^o)$ which equals $sin 30^o$ and $\sin 30^o = \frac{1}{2}$.

5. To show that $\sin(a+b) \cdot \sin(a-b) = \cos^2 b - \cos^2 a$, the sine identity for both the sum and the difference of angles must be used. Then the Pythagorean identity $\sin^2 \theta + \cos^2 \theta = 1$, or a form of this identity, will be applied.

$$
\begin{aligned}
\sin(a+b) \cdot \sin(a-b) &= \cos^2 b - \cos^2 a \\
(\sin a \cos b - \cos a \sin b)(\sin a \cos b + \cos a \sin b) &= \cos^2 b - \cos^2 a \rightarrow \text{multiply} \\
\sin^2 a \cos^2 b - \sin a \sin b \cos a \cos b + \sin a \sin b \cos a \cos b - \cos^2 a \sin^2 b &\rightarrow \text{simplify} \\
\sin^2 a \cos^2 b - \cos^2 a \sin^2 b &= \cos^2 b - \cos^2 a \rightarrow \sin^2 a = 1 - \cos^2 a \\
&\rightarrow \sin^2 b = 1 - \cos^2 b \\
(1 - \cos^2 a)\cos^2 b - \cos^2 a(1 - \cos^2 b) &= \cos^2 b - \cos^2 a \rightarrow \text{expand} \\
\cos^2 b - \cos^2 a \cos^2 b - \cos^2 a + \cos^2 ba \cos^2 b &= \cos^2 b - \cos^2 a \rightarrow \text{multiply} \\
\cos^2 b - \cos^2 a &= \cos^2 b - \cos^2 a
\end{aligned}
$$

6. To determine the value of $\tan(\pi + \theta)$ the tangent identity for the sum of angles must be applied. This identity is $\tan(a+b) = \frac{\tan a + \tan b}{1 - \tan a \tan b}$

$$
\begin{aligned}
\tan(a+b) &= \frac{\tan a + \tan b}{1 - \tan a \tan b} \\
\tan(\pi+\theta) &= \frac{\tan \pi + \tan \theta}{1 - \tan \pi \tan \theta} \\
\tan(\pi+\theta) &= \frac{(0) + \tan \theta}{1 - (0)\tan \theta} \\
\tan(\pi+\theta) &= \frac{\tan \theta}{1}
\end{aligned}
$$

7. To determine the exact value of tan 15^o , the tangent identity for the difference of angles must be used since no special angles have a sum of 15^o . However, 15^o is the difference between 45^o and 30^o . Therefore, $\tan(a-b) = \frac{\tan a - \tan b}{1 + \tan a \tan b}$ will be used.

$$\begin{aligned}
\tan(a-b) &= \frac{\tan a - \tan b}{1+\tan a \tan b}\\
\tan(45^o - 30^o) &= \frac{\tan 45^o - \tan 30^o}{1+\tan 45^o \tan 30^o}\\
\tan(45^o - 30^o) &= \frac{1-\frac{1}{\sqrt{3}}}{1+1\left(\frac{1}{\sqrt{3}}\right)} \rightarrow \text{simplify} \rightarrow \text{common deno min ator}\\
\tan(45^o - 30^o) &= \frac{\frac{\sqrt{3}}{\sqrt{3}}(1)-\frac{1}{\sqrt{3}}}{\frac{\sqrt{3}}{\sqrt{3}}(1)+\frac{1}{\sqrt{3}}} \rightarrow \text{simplify}\\
\tan(45^o - 30^o) &= \frac{\frac{\sqrt{3}-1}{\sqrt{3}}}{\frac{\sqrt{3}+1}{\sqrt{3}}} \rightarrow \text{divide}\\
\tan(45^o - 30^o) &= \left(\frac{\sqrt{3}-1}{\sqrt{3}}\right)\left(\frac{\sqrt{3}}{\sqrt{3}+1}\right) \rightarrow \text{simplify}\\
\tan(45^o - 30^o) &= \frac{\sqrt{3}-1}{\sqrt{3}+1} \rightarrow \text{rationalize deno min ator}\\
\tan(45^o - 30^o) &= \left(\frac{\sqrt{3}-1}{\sqrt{3}+1}\right)\left(\frac{\sqrt{3}-1}{\sqrt{3}-1}\right)\\
\tan(45^o - 30^o) &= \frac{\sqrt{9}-2\sqrt{3}+1}{\sqrt{9}-1} \rightarrow \text{simplify}\\
\tan(45^o - 30^o) &= \frac{3-2\sqrt{3}+1}{3-1}\\
\tan(45^o - 30^o) &= \frac{4-2\sqrt{3}}{2} \rightarrow \text{reduce fraction}\\
\tan(45^o - 30^o) &= 2-\sqrt{3}
\end{aligned}$$

8. To verify that $\sin \frac{\pi}{2} = 1$ the sine identity for the sum of angles will be used.

$$\begin{aligned}
\sin \frac{\pi}{2} &= \left(\sin \frac{\pi}{4} + \frac{\pi}{4}\right)\\
\sin(a+b) &= \sin a \cos b + \cos a \sin b\\
\sin\left(\frac{\pi}{4}+\frac{\pi}{4}\right) &= \sin \frac{\pi}{4} \cos \frac{\pi}{4} + \cos \frac{\pi}{4} \sin \frac{\pi}{4}\\
\sin\left(\frac{\pi}{4}+\frac{\pi}{4}\right) &= \left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right)+\left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right) \rightarrow \text{multiply}\\
\sin\left(\frac{\pi}{4}+\frac{\pi}{4}\right) &= \frac{1}{\sqrt{4}}+\frac{1}{\sqrt{4}} \rightarrow \text{simplify}\\
\sin\left(\frac{\pi}{4}+\frac{\pi}{4}\right) &= \frac{1}{2}+\frac{1}{2}\\
\sin\left(\frac{\pi}{4}+\frac{\pi}{4}\right) &= 1
\end{aligned}$$

9. To reduce $\cos(x+y)\cos y + \sin(x+y)\sin y$ to a single term requires the use of the cosine identity for the sum of angles and the sine identity for the sum of angles.

$$\begin{aligned}
&\cos(x+y)\cos y+\sin(x+y)\sin y \rightarrow \text{expand}\\
&\cos x\cos^2 y-\sin x\sin y\cos y+\sin x\sin y\cos y+\cos x\sin^2 y \rightarrow \text{simplify}\\
&\cos x\cos^2 y+\cos x\sin^2 y \rightarrow \text{remove common factor } (\cos x)\\
&\cos x(\cos^2 y+\sin^2 y) \rightarrow \sin^2 x+\cos^2 x=1\\
&\qquad\qquad\qquad\qquad \cos x(1)=\cos x\\
&\cos(x+y)\cos y+\sin(x+y)\sin y=\cos x
\end{aligned}$$

10. To solve $2\tan^2\left(x+\frac{\pi}{6}\right)-1=7$ for all values in the interval $[0,2\pi)$, the value of tan $\left(\frac{\pi}{6}\right)$ must be determined and then the tangent identity for the sum of angles must be applied to find the values within the stated interval.

$$\begin{aligned}
2\tan^2\left(x+\frac{\pi}{6}\right)-1&=7\\
2\tan^2\left(x+\frac{\pi}{6}\right)-1+1&=7+1\\
2\tan^2\left(x+\frac{\pi}{6}\right)&=8 \rightarrow \div \text{ both sides by 2}\\
\tan^2\left(x+\frac{\pi}{6}\right)&=4 \rightarrow \sqrt{\ } \text{both sides}\\
\sqrt{\tan^2\left(x+\frac{\pi}{6}\right)}&=\sqrt{4} \rightarrow \text{simplify}\\
\tan\left(x+\frac{\pi}{6}\right)&=2\\
\frac{\tan x+\tan\frac{\pi}{6}}{1-\tan x\tan\frac{\pi}{6}}&=2\\
\tan x+\tan\frac{\pi}{6}&=2(1-\tan x\tan\frac{\pi}{6})\\
\tan x+\frac{1}{\sqrt{3}}&=2(1-\tan x\frac{1}{\sqrt{3}}) \rightarrow \text{rationalize deno min ator}\\
\tan x+\left(\frac{\sqrt{3}}{\sqrt{3}}\right)\left(\frac{1}{\sqrt{3}}\right)&=2-\left(\frac{\sqrt{3}}{\sqrt{3}}\right)\left(\frac{\tan x}{\sqrt{3}}\right) \rightarrow \text{multiply}\\
\tan x+\frac{\sqrt{3}}{\sqrt{9}}&=2-\frac{\sqrt{3}\tan x}{\sqrt{9}} \rightarrow \text{simplify}\\
\tan x+\frac{\sqrt{3}}{\sqrt{9}}&=2-\frac{\sqrt{3}\tan x}{\sqrt{9}}\\
\tan x+\frac{\sqrt{3}}{3}&=2-\frac{\sqrt{3}}{\sqrt{3}} \rightarrow \text{common deno min ator (LS)}\\
\frac{3\tan x+\sqrt{3}\tan x}{3}&\approx 1.4226\\
1.5774\tan x&\approx 1.4226\\
\frac{1.5774\tan x}{1.5774}&\approx\frac{1.4226}{1.5774}\\
\tan x&\approx 0.9019\\
\tan^{-1}(\tan x)&\approx\tan^{-1}(0.9019)\\
x&\approx 0.7338\text{rad}
\end{aligned}$$

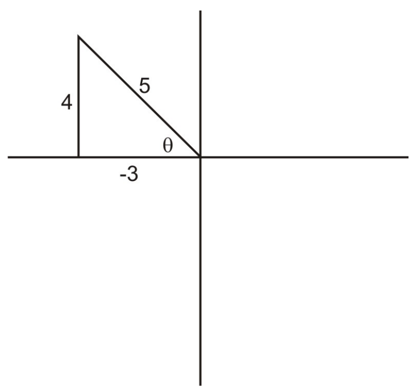
To determine the values, change the radians to degrees. The angles will be located in the 1st and 3rd quadrants.

$0.7338\left(\frac{180^o}{\pi}\right) \approx 42^o$. The angle in the 3rd quadrant would be approximately 222^o .

Double-Angle Identities

Review Exercises: Pages 260 – 265

1. If $\sin x = \frac{4}{5}$, and is the 2nd quadrant then



$$\begin{aligned}(h)^2 &= (s_1)^2+(s_2)^2\\(5)^2 &= (4)^2+(s_2)^2\\25 &= 16+(s_2)^2\\\sqrt{9} &= \sqrt{s^2}\\3 &= s\end{aligned}$$

In the 2nd quadrant, this value is negative.

For the above angle in standard position, $\cos x = -\frac{3}{5}$ and $x = -\frac{4}{3}$. The double-angle identities will be used to determine the exact values of $\sin 2x, \cos 2x, \tan 2x$.

$$\sin 2x = 2\sin x\cos x$$
$$\sin 2x = 2\left(\frac{4}{5}\right)\left(-\frac{3}{5}\right) \rightarrow \text{multiply}$$
$$\sin 2x = -\frac{24}{25}$$
$$\cos 2x = \cos^2 x - \sin^2 x$$
$$\cos 2x = \left(-\frac{3}{5}\right)^2 - \left(\frac{4}{5}\right)^2$$
$$\cos 2x = \frac{9}{25} - \frac{16}{25}$$
$$\cos 2x = -\frac{7}{25}$$
$$\tan 2x = \frac{2\tan x}{1-\tan^2 x}$$
$$\tan 2x = \frac{2\left(\frac{4}{3}\right)}{1-\left(-\frac{4}{3}\right)^2} \rightarrow \text{simplify}$$
$$\tan 2x = \frac{\frac{-8}{3}}{\frac{-7}{9}} \rightarrow \text{simplify}$$
$$\tan 2x = \left(\frac{-8}{3}\right)\left(\frac{-9}{7}\right)$$
$$\tan 2x = \frac{24}{7}$$

2. $\cos^2 15^\circ - \sin^2 15^\circ$ is the identity for $\cos 2a$.

$$\cos 2a = \cos^2 a - \sin^2 a$$

If $a = 15^\circ$ than $2a = 2(15^\circ) = 30^\circ$

$$\cos 30^\circ = \frac{\sqrt{3}}{2}$$

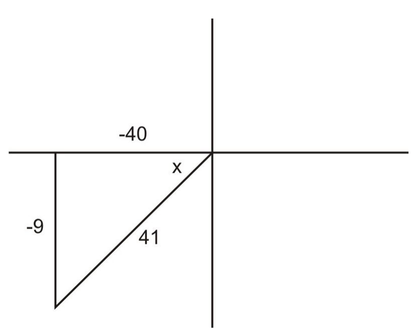
3. Verify: $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$. To verify this identity, the cosine identity for the sum of angles and the double-angle identities for cosine and sine will have to be applied.

$$
\begin{aligned}
\cos 3\theta &= 4\cos^3\theta - 3\cos\theta \\
\cos(a+b) &= \cos a\cos b - \sin a\sin b \rightarrow a = 2\theta; b = \theta \\
\cos(2\theta+\theta) &= (2\cos^2\theta - 1)\cos\theta - (2\sin\theta\cos\theta)\sin\theta \rightarrow \text{expand} \\
\cos(2\theta+\theta) &= 2\cos^3\theta - \cos\theta - 2\sin^2\theta\cos\theta \\
\cos(2\theta+\theta) &= \cos\theta(2\cos^2\theta - 1 - 2\sin^2\theta) \rightarrow \sin^2\theta + \cos^2\theta = 1 \\
\cos(2\theta+\theta) &= \cos\theta(2\cos^2\theta - 1 - 2(1-\cos^2\theta)) \rightarrow \text{simplify} \\
\cos(2\theta+\theta) &= \cos\theta(2\cos^2\theta - 1 - 2 + 2\cos^2\theta) \rightarrow \text{simplify} \\
\cos(2\theta+\theta) &= \cos\theta(4\cos^2\theta - 3) \rightarrow \text{expand} \\
\cos(2\theta+\theta) &= 4\cos^3\theta - 3\cos\theta
\end{aligned}
$$

4. Verify: $\sin 2t - \tan t = \tan t\cos 2t$. To verify this identity, the quotient identity for tangent must be used as well as the double-angle identities for sine and cosine.

$$
\begin{aligned}
\sin 2t - \tan t &= \tan t\cos 2t \rightarrow \sin 2t = 2\sin t\cos t \\
&\rightarrow \tan t = \frac{\sin t}{\cos t} \\
2\sin t\cos t - \frac{\sin t}{\cos t} &= \tan t\cos 2t \rightarrow \text{common denominator} \\
2\sin t\cos t\left(\frac{\cos t}{\cos t}\right) - \frac{\sin t}{\cos t} &= \tan t\cos 2t \rightarrow \text{simplify} \\
\frac{2\sin t\cos^2 t}{\cos t} - \frac{\sin t}{\cos t} &= \tan t\cos 2t \rightarrow \text{simplify} \\
\frac{2\sin t\cos^2 t - \sin t}{\cos t} &= \tan t\cos 2t \rightarrow \text{common factor } (\sin t) \\
\frac{(\sin t)2\cos^2 t - 1}{\cos t} &= \tan t\cos 2t \rightarrow \cos 2t = 2\cos^2 - 1 \\
\frac{(\sin t)\cos 2t}{\cos t} &= \tan t\cos 2t \rightarrow \cos 2t = 2\cos^2 t - 1 \\
\frac{(\sin t)\cos 2t}{\cos t} &= \tan t\cos 2t \rightarrow \tan t = \frac{\sin t}{\cos t} \\
\tan t\cos 2t &= \tan t\cos 2t
\end{aligned}
$$

5. If $\sin x = -\frac{9}{41}$ and is located in the 3rd quadrant, then:



$$
\begin{aligned}
(h)^2 &= (s_1)^2 + (s_2)^2 \\
(41)^2 &= (-9)^2 + (s_2)^2 \\
1681 &= 81 + (s_2)^2 \\
\sqrt{1600} &= \sqrt{s^2} \\
40 &= s
\end{aligned}
$$

In the 3rd quadrant this value is negative.

For the above angle in standard position, $\cos x = -\frac{40}{41}$ and $\tan x = \frac{9}{40}$. The double-angle identities will be used to determine the exact values of $\sin 2x, \cos 2x, \tan 2x$.

$$
\begin{aligned}
\sin 2x &= 2 \sin x \cos x \\
\sin 2x &= 2\left(-\frac{9}{41}\right)\left(-\frac{40}{41}\right) \rightarrow \text{multiply} \\
\sin 2x &= \frac{720}{1681} \\
\cos 2x &= 2\cos^2 x - 1 \\
\cos 2x &= 2\left(-\frac{40}{41}\right)^2 - 1 \\
\cos 2x &= \frac{3200}{1681} - 1 \rightarrow \text{common denominator} \\
\cos 2x &= \frac{3200}{1681} - 1\left(\frac{1681}{1681}\right) \rightarrow \text{simplify} \\
\cos 2x &= \frac{1519}{1681} \\
\tan 2x &= \frac{\sin 2x}{\cos 2x} \rightarrow \sin 2x = \frac{720}{1681}; \cos 2x = \frac{1519}{1681} \\
\tan 2x &= \frac{\frac{720}{1681}}{\frac{1591}{1681}} \rightarrow \text{simplify} \\
\tan 2x &= \left(\frac{720}{1681}\right)\left(\frac{1681}{1519}\right) \\
\tan 2x &= \frac{720}{1519}
\end{aligned}
$$

6. To find all the solutions for x in the equation $\sin 2x + \sin x = 0$ such that $0 \leq x < 2\pi$, the double-angle identity for sine must be used.

$$\begin{aligned}
\sin 2x + \sin x = 0 &\rightarrow \sin 2x = 2\sin x \cos x \\
2\sin x \cos x + \sin x = 0 &\rightarrow \text{common factor } (\sin x) \\
(\sin x) 2\cos x + 1 &= 0 \\
\text{Then } \sin x = 0 &\text{ or } 2\cos x + 1 = 0 \rightarrow \text{solve} \\
\sin^{-1}(\sin x) &= \sin^{-1}(0) \\
x &= 0, \pi \\
2\cos x + 1 &= 0 \\
\cos x &= -\frac{1}{2} \\
\cos^{-1}(\cos x) &= \cos^{-1}\left(-\frac{1}{2}\right) \\
x &= \frac{2\pi}{3}, \frac{4\pi}{3}
\end{aligned}$$

7. To find all the solutions for x in the equation $\cos^2 x - \cos 2x = 0$ such that $0 \le x < 2\pi$, the double-angle identity for cosine must be used.

$$\begin{aligned}
\cos^2 x - \cos 2x = 0 &\rightarrow \cos 2x = 2\cos^2 x - 1 \\
\cos^2 x - (2\cos^2 x - 1) = 0 &\rightarrow \text{simplify} \\
-\cos^2 x + 1 = 0 &\rightarrow \div \text{ by } -1 \\
\cos^2 x - 1 = 0 &\rightarrow \text{factor} \\
(\cos x + 1)(\cos x - 1) = 0 &\rightarrow \text{solve} \\
\text{Then } \cos x + 1 = 0 &\text{ or } & \cos x - 1 &= 0 \\
\cos x &= -1 & \cos x &= 1 \\
\cos^{-1}(\cos x) &= \cos^{-1}(-1) & \cos^{-1}(\cos x) &= \cos^{-1}(1) \\
x &= \pi & x &= 0
\end{aligned}$$

8. The formula for $\cos^2 x$ in terms of the first power of cosine is $\cos^2 x = \frac{1}{2}(\cos 2x + 1)$. To express $\cos^4 x$ in terms of the first power of cosine, the first step is to realize that $cos^4 x = (cos^2 x)^2$. Therefore:

$$\cos^2 x = \frac{1}{2}(\cos 2x + 1) \rightarrow \text{square both sides}$$
$$(\cos^2 x)^2 = \left[\frac{1}{2}(\cos 2x + 1)\right]^2 \rightarrow \text{expand}$$
$$\cos^2 x = \frac{1}{4}(\cos^2 2x + 2\cos 2x + 1)$$
$$\text{if } \cos^2 x = \frac{1}{2}(\cos 2x + 1) \rightarrow \text{replace } x \text{ with } 2x \text{ then } \cos^2 2x = \frac{1}{2}(\cos 4x + 1)$$
$$\cos^4 x = \frac{1}{4}\left[\frac{1}{2}(\cos 4x + 1) + 2\cos 2x + 1\right] \rightarrow \text{expand}$$
$$\cos^4 x = \frac{1}{4}\left[\frac{1}{2}\cos 4x + \frac{1}{2} + 2\cos 2x + 1\right] \rightarrow \text{simplify}$$
$$\cos^4 x = \frac{1}{4}\left[\frac{1}{2}\cos 4x + 2\cos 2x + \frac{3}{2}\right] \rightarrow \text{multiply}$$
$$\cos^4 x = \frac{1}{8}\cos 4x + \frac{2}{4}\cos 2x + \frac{3}{8} \rightarrow \text{common denominator}$$
$$\cos^4 x = \frac{1}{8}\cos 4x + \left(\frac{2}{2}\right)\frac{2}{4}\cos 2x + \frac{3}{8}$$
$$\cos^4 x = \frac{1}{8}\cos 4x + \frac{4}{8}\cos 2x + \frac{3}{8} \rightarrow \text{simplify}$$
$$\cos^4 x = \frac{\cos 4x + 4\cos 2x + 3}{8}$$

9. The formula for $sin^2 x$ in terms of the first power of cosine is $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$. To express $sin^4 x$ in terms of the first power of cosine, the first step is to realize that $\sin^4 x = (\sin^2 x)^2$. Therefore:

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x) \rightarrow \text{square both sides}$$
$$(\sin^2 x)^2 = \left[\frac{1}{2}(1 - \cos 2x)\right]^2 \rightarrow \text{expand}$$
$$\sin^2 x = \frac{1}{4}(1 - 2\cos 2x + \cos^2 2x)$$
$$\text{if } \sin^2 x = \frac{1}{2}(1 - \cos 2x) \rightarrow \text{replace } x \text{ with } 2x \text{ then } \cos^2 2x = \frac{1}{2}(\cos 4x + 1)$$
$$\sin^4 x = \frac{1}{4}\left[1 - 2\cos 2x + \frac{1}{2}(\cos 4x + 1)\right] \rightarrow \text{expand}$$
$$\sin^4 x = \frac{1}{4}\left[1 - 2\cos 2x + \frac{1}{2}\cos 4x + \frac{1}{2}\right] \rightarrow \text{simplify}$$
$$\sin^4 x = \frac{1}{4}\left[\frac{3}{2} - 2\cos 2x + \frac{1}{2}\cos 4x\right] \rightarrow \text{multiply}$$
$$\sin^4 x = \frac{3}{8} - \frac{2}{4}\cos 2x + \frac{1}{8}\cos 4x \rightarrow \text{common denominator}$$
$$\sin^4 x = \frac{3}{8} - \left(\frac{2}{2}\right)\frac{2}{4}\cos 2x + \frac{1}{8}\cos 4x$$
$$\sin^4 x = \frac{3}{8} - \frac{4}{8}\cos 2x + \frac{1}{8}\cos 4x \rightarrow \text{simplify}$$
$$\sin^4 x = \frac{3 - 4\cos 2x + \cos 4x}{8}$$

10. a) To rewrite $sin^2 x \cos^2 2x$ in terms of the first power of cosine, determine the product by using the formulas: $\sin^2 x = \frac{1}{2}(1 - cos 2x)$ and $\cos^2 2x = \frac{1}{2}(\cos 4x + 1)$

$$
\begin{aligned}
&\sin^2 x \cos^2 2x \\
&\left[\frac{1}{2}(1-\cos 2x)\right]\left[\frac{1}{2}(\cos 4x+1)\right] \rightarrow \text{expand} \\
&\left(\frac{1}{2}-\frac{1}{2}\cos 2x\right)\left(\frac{1}{2}\cos 4x+\frac{1}{2}\right) \rightarrow \text{expand} \\
&\frac{1}{4}\cos 4x+\frac{1}{4}-\frac{1}{4}\cos 2x\cos 4x-\frac{1}{4}\cos 2x \rightarrow \text{common factor}\left(\frac{1}{4}\right) \\
&\frac{1}{4}(\cos 4x+1-\cos 2x\cos 4x-\cos 2x) \rightarrow \text{rearrange} \\
&\frac{1}{4}(1-\cos 2x+\cos 4x-\cos 2x\cos 4x)
\end{aligned}
$$

b) To rewrite $tan^4 2x$ in terms of the first power of cosine, the quotient identity for tangent will be used along with $\cos^4 x = \frac{\cos 4x + 4\cos 2x + 3}{8}$ and $\sin^4 x = \frac{3 - 4\cos 2x + \cos 4x}{8}$.

$$
\begin{aligned}
\tan^4 x &= \frac{\sin^4 x}{\cos^4 x} \\
\tan^4 2x &= \frac{\sin^4 2x}{\cos^4 2x} \\
\tan^4 x &= \frac{\frac{3-4\cos 2x+\cos 4x}{8}}{\frac{3+4\cos 2x+\cos 4x}{8}} \rightarrow \text{replace } x \text{ with } 2x \\
\tan^4 x &= \frac{\frac{3-4\cos 4x+\cos 8x}{8}}{\frac{3+4\cos 4x+\cos 8x}{8}} \rightarrow \text{simplify} \\
\tan^4 2x &= \left(\frac{3-4\cos 4x+\cos 8x}{8}\right)\left(\frac{8}{3+4\cos 4x+\cos 8x}\right) \rightarrow \text{multiply} \\
\tan^4 2x &= \left(\frac{3-4\cos 4x+\cos 8x}{3+4\cos 4x+\cos 8x}\right)
\end{aligned}
$$

Half-Angle Identities

Review Exercises:

1. To determine the exact value of $\cos 112.5^\circ$, the angle must be expressed in the form of a half-angle. Once this is done, the half-angle identity for cosine, $\cos\frac{\theta}{2} = \pm\sqrt{\frac{\cos\theta + 1}{2}}$ can be used to determine the exact value.

$$\cos 112.5^\circ = \cos \frac{225^\circ}{2}$$
$$\cos \frac{\theta}{2} = \pm\sqrt{\frac{\cos\theta + 1}{2}}$$
$$\cos \frac{225^\circ}{2} = \pm\sqrt{\frac{\cos 225^\circ + 1}{2}}$$
$$\cos \frac{225^\circ}{2} = \pm\sqrt{\frac{\frac{-1}{\sqrt{2}} + 1}{2}} \rightarrow \text{(common denominator)}$$
$$\cos \frac{225^\circ}{2} = \pm\sqrt{\frac{\frac{1}{\sqrt{2}} + \left(\frac{\sqrt{2}}{\sqrt{2}}\right)1}{2}} \rightarrow \text{simplify}$$
$$\cos \frac{225^\circ}{2} = \pm\sqrt{\frac{\frac{-1}{\sqrt{2}} + \frac{\sqrt{2}}{\sqrt{2}}}{2}} \rightarrow \text{simplify}$$
$$\cos \frac{225^\circ}{2} = \pm\sqrt{\frac{\frac{-1+\sqrt{2}}{\sqrt{2}}}{2}} \rightarrow \text{simplify}$$
$$\cos \frac{225^\circ}{2} = \pm\sqrt{\left(\frac{-1+\sqrt{2}}{\sqrt{2}}\right)\left(\frac{1}{2}\right)} \rightarrow \text{simplify}$$
$$\cos \frac{225^\circ}{2} = \pm\sqrt{\left(\frac{-1+\sqrt{2}}{2\sqrt{2}}\right)} \rightarrow \text{rationalize denominator}$$
$$\cos \frac{225^\circ}{2} = \pm\sqrt{\left(\frac{-1+\sqrt{2}}{2\sqrt{2}}\right)\left(\frac{\sqrt{2}}{\sqrt{2}}\right)} \rightarrow \text{simplify}$$
$$\cos \frac{225^\circ}{2} = \pm\sqrt{\left(\frac{-\sqrt{2}+\sqrt{4}}{2\sqrt{4}}\right)} \rightarrow \text{simplify}$$
$$\cos \frac{225^\circ}{2} = \pm\sqrt{\left(\frac{-\sqrt{2}+2}{4}\right)} \rightarrow \text{simplify}$$

$$\cos \frac{225^\circ}{2} = \pm\frac{\sqrt{-\sqrt{2}+2}}{2}$$

112.5° is an angle located in the 2^{nd} quadrant. The cosine of an angle in this quadrant is negative. The exact value of this angle is:

$$\cos \frac{225^\circ}{2} = -\frac{\sqrt{-\sqrt{2}+2}}{2}$$

2. To determine the exact value of 105° , the angle must be expressed in the form of a half-angle. Once this is done, the half-angle identity for sine, $\sin \frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{2}}$ can be used to determine the exact value.

$$\sin 105^\circ = \sin \frac{210^\circ}{2}$$
$$\sin \frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{2}}$$
$$\sin \frac{210^\circ}{2} = \pm\sqrt{\frac{1-\cos 210^\circ}{2}}$$
$$\sin \frac{210^\circ}{2} = \pm\sqrt{\frac{1-\left(-\frac{\sqrt{3}}{2}\right)}{2}} \rightarrow \text{common denominator}$$
$$\sin \frac{210^\circ}{2} = \pm\sqrt{\frac{1\left(\frac{2}{2}\right)-\left(-\frac{\sqrt{3}}{2}\right)}{2}} \rightarrow \text{simplify}$$
$$\sin \frac{210^\circ}{2} = \pm\sqrt{\frac{\frac{2}{2}+\left(\frac{\sqrt{3}}{2}\right)}{2}} \rightarrow \text{simplify}$$
$$\sin \frac{210^\circ}{2} = \pm\sqrt{\frac{\frac{2+\sqrt{3}}{2}}{2}} \rightarrow \text{simplify}$$
$$\sin \frac{210^\circ}{2} = \pm\sqrt{\left(\frac{2+\sqrt{3}}{2}\right)\left(\frac{1}{2}\right)} \rightarrow \text{simplify}$$
$$\sin \frac{210^\circ}{2} = \pm\sqrt{\left(\frac{2+\sqrt{3}}{2}\right)} \rightarrow \text{simplify}$$

$$\sin \frac{210^\circ}{2} = \pm\frac{\sqrt{2+\sqrt{3}}}{2}$$

105° is an angle located in the 2^{nd} quadrant. The sine of an angle in this quadrant is positive. The exact value of this angle is:

$$\sin \frac{210^\circ}{2} = \frac{\sqrt{2+\sqrt{3}}}{2}$$

3. To determine the exact value of $\tan \frac{7\pi}{8}$, the angle must be expressed in the form of a half-angle. Once this is done, the half-angle identity for tangent, $\tan \frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}$ can be used to determine the exact value.

$$\tan\frac{7\pi}{8} = \tan\frac{\frac{7\pi}{4}}{2}$$

$$\tan\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}$$

$$\tan\frac{\frac{7\pi}{4}}{2} = \pm\sqrt{\frac{1-\cos\frac{7\pi}{4}}{1+\cos\frac{7\pi}{4}}}$$

$$\tan\frac{\frac{7\pi}{4}}{2} = \pm\sqrt{\frac{1-\frac{1}{\sqrt{2}}}{1+\frac{1}{\sqrt{2}}}} \rightarrow \text{common denominator}$$

$$\tan\frac{\frac{7\pi}{4}}{2} = \pm\sqrt{\frac{\left(\frac{\sqrt{2}}{\sqrt{2}}\right)1-\frac{1}{\sqrt{2}}}{\left(\frac{\sqrt{2}}{\sqrt{2}}\right)1+\frac{1}{\sqrt{2}}}} \rightarrow \text{simplify}$$

$$\tan\frac{\frac{7\pi}{4}}{2} = \pm\sqrt{\frac{\left(\frac{\sqrt{2}}{\sqrt{2}}\right)-\frac{1}{\sqrt{2}}}{\left(\frac{\sqrt{2}}{\sqrt{2}}\right)+\frac{1}{\sqrt{2}}}} \rightarrow \text{simplify}$$

$$\tan\frac{\frac{7\pi}{4}}{2} = \pm\sqrt{\frac{\left(\frac{\sqrt{2}-1}{\sqrt{2}}\right)}{\left(\frac{\sqrt{2}+1}{\sqrt{2}}\right)}} \rightarrow \text{simplify}$$

$$\tan\frac{\frac{7\pi}{4}}{2} = \pm\sqrt{\left(\frac{\sqrt{2}-1}{\sqrt{2}}\right)\left(\frac{\sqrt{2}}{\sqrt{2}+1}\right)} \rightarrow \text{simplify}$$

$$\tan\frac{\frac{7\pi}{4}}{2} = \pm\sqrt{\left(\frac{\sqrt{4}-\sqrt{2}}{\sqrt{4}+\sqrt{2}}\right)} \rightarrow \text{simplify}$$

$$\tan\frac{\frac{7\pi}{4}}{2} = \pm\sqrt{\left(\frac{2-\sqrt{2}}{2+\sqrt{2}}\right)\left(\frac{2-\sqrt{2}}{2-\sqrt{2}}\right)} \rightarrow \text{rationalize denominator}$$

$$\tan\frac{\frac{7\pi}{4}}{2} = \pm\sqrt{\left(\frac{4-4\sqrt{2}+\sqrt{4}}{4-\sqrt{4}}\right)} \rightarrow \text{simplify}$$

$$\tan\frac{\frac{7\pi}{4}}{2} = \pm\sqrt{\left(\frac{4-4\sqrt{2}+2}{4-2}\right)} \rightarrow \text{simplify}\tan\frac{\frac{7\pi}{4}}{2} \qquad = \pm\sqrt{\left(\frac{6-4\sqrt{2}}{2}\right)} \rightarrow \text{simplify}$$

$$\tan\frac{\frac{7\pi}{4}}{2} = \pm\sqrt{3-2\sqrt{2}}$$

$\frac{7\pi}{8}$ is an angle located in the 2nd quadrant. The tangent of an angle in this quadrant is negative. The exact value of this angle is:

$$\tan\frac{\frac{7\pi}{4}}{2} = \pm\sqrt{3-2\sqrt{2}}$$

4. To determine the exact value of $\tan \frac{\pi}{8}$, the angle must be expressed in the form of a half-angle. Once this is done, the half-angle identity for tangent, $\tan \frac{\theta}{2} = \pm \sqrt{\frac{1-\cos\theta}{1+\cos\theta}}$ can be used to determine the exact value.

$$\tan \frac{\pi}{8} = \tan \frac{\frac{\pi}{4}}{2}$$

$$\tan \frac{\theta}{2} = \pm \sqrt{\frac{1-\cos\theta}{1+\cos\theta}}$$

$$\tan \frac{\frac{\pi}{4}}{2} = \pm \sqrt{\frac{1-\cos\frac{\pi}{4}}{1+\cos\frac{\pi}{4}}}$$

$$\tan \frac{\frac{\pi}{4}}{2} = \pm \sqrt{\frac{1-\frac{1}{\sqrt{2}}}{1+\frac{1}{\sqrt{2}}}} \rightarrow \text{common denominator}$$

$$\tan \frac{\frac{\pi}{4}}{2} = \pm \sqrt{\frac{\left(\frac{\sqrt{2}}{\sqrt{2}}\right)1-\frac{1}{\sqrt{2}}}{\left(\frac{\sqrt{2}}{\sqrt{2}}\right)1-\frac{1}{\sqrt{2}}}} \rightarrow \text{simplify}$$

$$\tan \frac{\frac{\pi}{4}}{2} = \pm \sqrt{\frac{\frac{\sqrt{2}-1}{\sqrt{2}}}{\frac{\sqrt{2}+1}{\sqrt{2}}}} \rightarrow \text{simplify}$$

$$\tan \frac{\frac{\pi}{4}}{2} = \pm \sqrt{\left(\frac{\sqrt{2}-1}{\sqrt{2}}\right)\left(\frac{\sqrt{2}}{\sqrt{2}+1}\right)} \rightarrow \text{simplify}$$

$$\tan \frac{\frac{\pi}{4}}{2} = \pm \sqrt{\left(\frac{\sqrt{4}-\sqrt{2}}{\sqrt{4}+\sqrt{2}}\right)} \rightarrow \text{simplify}$$

$$\tan \frac{\frac{\pi}{4}}{2} = \pm \sqrt{\left(\frac{2-\sqrt{2}}{2+\sqrt{2}}\right)} \rightarrow \text{rationalize denominator}$$

$$\tan \frac{\frac{\pi}{4}}{2} = \pm \sqrt{\left(\frac{2-\sqrt{2}}{2+\sqrt{2}}\right)\left(\frac{2-\sqrt{2}}{2-\sqrt{2}}\right)} \rightarrow \text{simplify}$$

$$\tan \frac{\frac{\pi}{4}}{2} = \pm \sqrt{\left(\frac{4-4\sqrt{2}+\sqrt{4}}{4-\sqrt{4}}\right)} \rightarrow \text{simplify}$$

$$\tan \frac{\frac{\pi}{4}}{2} = \pm \sqrt{\left(\frac{4-4\sqrt{2}+2}{4-2}\right)} \rightarrow \text{simplify}$$

$$\tan \frac{\frac{\pi}{4}}{2} = \pm \sqrt{\left(\frac{6-4\sqrt{2}}{2}\right)} \rightarrow \text{simplify}$$

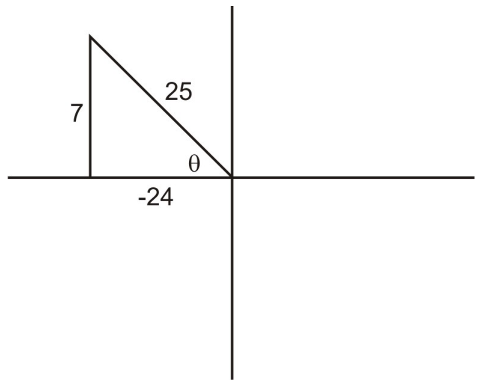
$$\tan \frac{\frac{\pi}{4}}{2} = \pm \sqrt{3-2\sqrt{2}}$$

is an angle located in the 1st quadrant. The tangent of an angle in this quadrant is positive. The exact value of this angle is:

$\frac{\pi}{8}$ is an angle located in the 1st quadrant. The tangent of an angle in this quadrant is positive. The exact value of this angle is:

$$\tan\frac{\frac{\pi}{4}}{2} = \pm\sqrt{3-2\sqrt{2}}$$

5. If $\sin\theta = \frac{7}{25}$ and is located in the 2nd quadrant, then:



$$\begin{aligned}(h)^2 &= (s_1)^2+(s_2)^2\\ (25)^2 &= (7)^2+(s_2)^2\\ 625 &= 49+(s_2)^2\\ \sqrt{576} &= \sqrt{s^2}\\ 24 &= s\end{aligned}$$

In the 2nd quadrant this value is negative and $\cos\theta = -\frac{24}{25}$.

TABLE 3.1:

Steps	$\sin\frac{\theta}{2}$	$\cos\frac{\theta}{2}$	$\tan\frac{\theta}{2}$
=	$\pm\sqrt{\frac{1-\cos\theta}{2}}$	$\pm\sqrt{\frac{\cos\theta+1}{2}}$	$\pm\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}$
$\cos\theta = -\frac{24}{25}$	$\pm\sqrt{\frac{1-(-\frac{24}{25})}{2}}$	$\pm\sqrt{\frac{(-\frac{24}{25})+1}{2}}$	$\pm\sqrt{\frac{1-(-\frac{24}{25})}{1+(-\frac{24}{25})}}$
simplify	$\pm\sqrt{\frac{1+\frac{24}{25}}{2}}$	$\pm\sqrt{\frac{-\frac{24}{25}+1}{2}}$	$\pm\sqrt{\frac{1+\frac{24}{25}}{1-\frac{24}{25}}}$
Common denominator	$\pm\sqrt{\frac{\frac{25}{25}+\frac{24}{25}}{2}}$	$\pm\sqrt{-\frac{\frac{24}{25}+\frac{25}{25}}{2}}$	$\pm\sqrt{-\frac{\frac{24}{25}+\frac{25}{25}}{2}\frac{\frac{24}{25}-\frac{25}{25}}{2}}$
simplify	$\pm\sqrt{\left(\frac{49}{25}\right)\left(\frac{1}{2}\right)}$	$\pm\sqrt{\left(\frac{1}{25}\right)\left(\frac{1}{2}\right)}$	$\pm\sqrt{\frac{\frac{49}{25}}{\frac{1}{25}}} = \pm\sqrt{\left(\frac{49}{25}\right)\left(\frac{25}{1}\right)}$
simplify	$\pm\sqrt{\frac{49}{50}}$	$\pm\sqrt{\frac{1}{50}}$	$\pm\sqrt{\frac{49}{1}}$

TABLE 3.1: (continued)

Steps	$\sin\frac{\theta}{2}$	$\cos\frac{\theta}{2}$	$\tan\frac{\theta}{2}$
simplify $\sqrt{25\cdot 2}=5\sqrt{2}$	$\pm\frac{7}{5\sqrt{2}}$	$\pm\frac{1}{5\sqrt{2}}$	$\pm\frac{7}{1}$
Rationalize denominator	$\pm\frac{7}{5\sqrt{2}}\left(\sqrt{\frac{2}{2}}\right)=\pm\frac{7\sqrt{2}}{10}$	$\pm\frac{1}{5\sqrt{2}}\left(\sqrt{\frac{2}{2}}\right)=\pm\frac{\sqrt{2}}{10}$	$\pm\frac{7}{1}$
2 nd *quadrantangle*	$\sin\frac{\theta}{2}=\frac{7\sqrt{2}}{10}$	$\cos\frac{\theta}{2}=-\frac{\sqrt{2}}{10}$	$\tan\frac{\theta}{2}=-7$

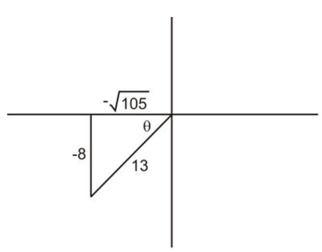
6. To verify the identity $\tan\frac{b}{2}=\frac{\sec b}{\sec b\csc b+\csc b}$, the half-angle identity for tangent must be used as well as the reciprocal identities for secant and cosecant.

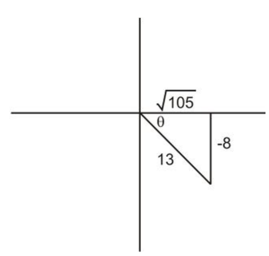
$$\tan\frac{b}{2}=\frac{\sec b}{\sec b\csc b+\csc b}\to\text{common factor }(\csc b)$$
$$\tan\frac{b}{2}=\frac{\sec b}{\csc b(\sec b+1)}\to\text{reciprocal identities}$$
$$\tan\frac{b}{2}=\frac{\frac{1}{\cos b}}{\frac{1}{\sin b}\left(\frac{1}{\cos b}+1\right)}\to\text{multiply}$$
$$\tan\frac{b}{2}=\frac{\frac{1}{\cos b}}{\frac{1}{\sin b\cos b}+\frac{1}{\sin b}}\to\text{common denominator}$$
$$\tan\frac{b}{2}=\frac{\frac{1}{\cos b}}{\frac{1}{\sin b\cos b}+\left(\frac{\cos b}{\cos b}\right)\frac{1}{\sin b}}\to\text{simplify}$$
$$\tan\frac{b}{2}=\frac{\frac{1}{\cos b}}{\frac{1+\cos b}{\sin b\cos b}}\to\text{simplify}$$
$$\tan\frac{b}{2}=\left(\frac{1}{\cos b}\right)\left(\frac{\sin b\cos b}{1+\cos b}\right)\to\text{simplify}$$
$$\tan\frac{b}{2}=\left(\frac{1}{\cancel{\cos b}}\right)\left(\frac{\sin b\cancel{\cos b}}{1+\cos b}\right)\to\text{simplify}$$
$$\tan\frac{b}{2}=\frac{\sin b}{1+\cos b}\text{ and }\tan\frac{b}{2}=\pm\sqrt{\frac{1-\cos b}{1+\cos b}}\to\text{half - angle identity for tan gent}$$
$$\therefore\pm\sqrt{\frac{1-\cos b}{1+\cos b}}=\frac{\sin b}{1+\cos b}\to\text{square both sides}$$
$$\therefore\left(\pm\sqrt{\frac{1-\cos b}{1+\cos b}}\right)^2=\left(\frac{\sin b}{1+\cos b}\right)^2\to\text{expand}$$
$$\frac{1-\cos b}{1+\cos b}=\frac{\sin^2 b}{(1+\cos b)^2}$$
$$(1-\cos b)(1+\cos b)(1+\cos b)=\sin^2 b(1+\cos b)\to\text{common factor}$$
$$\frac{(1-\cos b)(1+\cos b)\cancel{(1+\cos b)}}{\cancel{(1+\cos b)}}=\frac{\sin^2 b\cancel{(1+\cos b)}}{\cancel{(1+\cos b)}}\to\text{simplify}$$
$$(1-\cos b)(1+\cos b)=\sin^2 b\to\text{multiply}$$
$$1-\cos^2 b=\sin^2 b\to\sin^2 b+\cos^2 b=1$$
$$\sin^2 b=\sin^2 b$$

7. To verify the identity $\cot \frac{c}{2} = \frac{\sin c}{1-\cos c}$, the quotient identity for cotangent must be applied as well as the half-angle identities for sine and cosine.

$$\begin{aligned}
\cot \frac{c}{2} &= \frac{\sin c}{1-\cos c} \rightarrow \cot\theta = \frac{\cos\theta}{\sin\theta} \\
\frac{\cos \frac{c}{2}}{\sin \frac{c}{2}} &= \frac{\sin c}{1-\cos c} \rightarrow \text{half - angle identities} \\
\frac{\pm\sqrt{\frac{\cos c+1}{2}}}{\pm\sqrt{\frac{1-\cos c}{2}}} &= \frac{\sin c}{1-\cos c} \rightarrow \text{simplify (LS)} \\
\pm\sqrt{\left(\frac{\cos c+1}{2}\right)\left(\frac{2}{1-\cos c}\right)} &= \frac{\sin c}{1-\cos c} \rightarrow \text{simplify} \\
\pm\sqrt{\left(\frac{\cos c+1}{\cancel{2}}\right)\left(\frac{\cancel{2}}{1-\cos c}\right)} &= \frac{\sin c}{1-\cos c} \rightarrow \text{square both sides} \\
\left(\pm\sqrt{frac\cos c+11-\cos c}\right)^2 &= \left(\frac{\sin c}{1-\cos c}\right)^2 \rightarrow \text{simplify} \\
\frac{\cos c+1}{1-\cos c} &= \frac{\sin^2 c}{(1-\cos c)^2} \rightarrow \text{expand} \\
(\cos c+1)(1-\cos c)(1-\cos c) &= \sin^2 c(1-\cos c) \rightarrow \text{common factor} \\
\frac{(\cos c+1)(1-\cos c)\cancel{(1-\cos c)}}{\cancel{(1-\cos c)}} &= \frac{\sin^2 c\cancel{(1-\cos c)}}{\cancel{(1-\cos c)}} \rightarrow \text{simplify} \\
(\cos c+1)(1-\cos c) &= \sin^2 c \rightarrow \text{multiply} \\
1-\cos^2 c &= \sin^2 c \rightarrow \sin^2 c+\cos^2 c = 1 \\
\sin^2 c &= \sin^2 c
\end{aligned}$$

8. If $\sin u = -\frac{8}{13}$, the angle must be located in either the 3rd or 4th quadrant since the sine function is negative here.





$$\begin{aligned}
(h)^2 &= (s_1)^2+(s_2)^2 \\
(13)^2 &= (-8)^2+(s_2)^2 \\
169 &= 64+(s_2)^2 \\
\sqrt{105} &= \sqrt{s^2} \\
\sqrt{105} &= s
\end{aligned}$$

$-\sqrt{105}$ is inadmissible in the half-angle formula. Therefore the angle is in the 4th quadrant and $\cos u = \frac{\sqrt{105}}{13}$

$$\cos\frac{u}{2} = \pm\sqrt{\frac{\cos u + 1}{2}}$$
$$\cos\frac{u}{2} = \pm\sqrt{\frac{\frac{\sqrt{105}}{13} + 1}{2}} \rightarrow \text{common denominator}$$
$$\cos\frac{u}{2} = \pm\sqrt{\frac{\frac{\sqrt{105}}{13} + \left(\frac{13}{13}\right)1}{2}} \rightarrow \text{simplify}$$
$$\cos\frac{u}{2} = \pm\sqrt{\frac{\frac{\sqrt{105}+13}{13}}{2}} \rightarrow \text{simplify}$$
$$\cos\frac{u}{2} = \pm\sqrt{\left(\frac{\sqrt{105}+13}{13}\right)\left(\frac{1}{2}\right)} \rightarrow \text{simplify}$$
$$\cos\frac{u}{2} = \sqrt{\frac{\sqrt{105}+13}{26}}$$

The angle is located in the 4th quadrant where the cosine function has a positive value.

9. To solve the trigonometric equation $2\cos^2\frac{x}{2} = 1$ for values of x such that $0 \leq x < 2\pi$ the half-angle identity for cosine must be used.

$$2\cos^2\frac{x}{2} = 1 \rightarrow \div \text{ both sides by 2}$$
$$\cos^2\frac{x}{2} = \frac{1}{2} \rightarrow \text{half - angle identity}$$
$$\left(\pm\sqrt{\frac{\cos x + 1}{2}}\right)^2 = \frac{1}{2} \rightarrow \text{simplify}$$
$$\frac{\cos x + 1}{2} = \frac{1}{2} \rightarrow \text{simplify}$$
$$2(\cos x + 1) = 2 \rightarrow \text{multiply}$$
$$2\cos x + 2 = 2 \rightarrow \text{solve}$$
$$2\cos x + 2 - 2 = 2 - 2$$
$$\frac{2\cos x}{2} = \frac{0}{2}$$
$$\cos x = 0$$
$$\cos^{-1}(\cos x) = \cos^{-1}(0)$$
$$x = \frac{\pi}{2} \text{ and } \frac{3\pi}{2}$$

10. To solve the trigonometric equation $\tan\frac{a}{2} = 4$ for all values of x such that $0^\circ \leq x < 360^\circ$, the half-angle identity for tangent must be used.

$$\tan\frac{a}{2} = 4 \rightarrow \text{half - angle identity}$$
$$\left(\pm\sqrt{\frac{1-\cos a}{1+\cos a}}\right) = 4 \rightarrow \text{square both sides}$$
$$\left(\pm\sqrt{\frac{1-\cos a}{1+\cos a}}\right)^2 = (4)^2 \rightarrow \text{simplify}$$
$$\frac{1-\cos a}{1+\cos a} = 16$$
$$16(1+\cos a) = 1-\cos a \rightarrow \text{multiply}$$
$$16+16\cos a = 1-\cos a \rightarrow \text{solve}$$
$$16-16+16\cos a = 1-\cos a-16$$
$$16\cos a = -\cos a-15$$
$$16\cos a+\cos a = -\cos a+\cos a-15$$
$$\frac{17\cos a}{17} = \frac{-15}{17}$$
$$\cos a = -\frac{15}{17} \rightarrow \text{use calculator}$$
$$\cos^{-1}(\cos a) = \cos^{-1}\left(-\frac{15}{17}\right)$$

The cosine function has a negative value in both the 2^{nd} and 3^{rd} quadrants. There are 2 values for angle a .

$$a \approx 152^\circ \text{ and } a \approx 108^\circ$$

Product-and Sum, Sum-and-Product and Linear Combinations of Identities

Review Exercises:

1. To express $\sin 9x+\sin 5x$ as a product, the sum to product formula for sine must be used.

$$\sin\alpha+\sin\beta = 2\sin\left(\frac{\alpha+\beta}{2}\right)\cdot\cos\left(\frac{\alpha-\beta}{2}\right) \rightarrow \alpha = 9x$$
$$\sin 9x+\sin 5x = 2\sin\left(\frac{9x+5x}{2}\right)\cdot\cos\left(\frac{9x-5x}{2}\right) \rightarrow \text{simplify}$$
$$\sin 9x+\sin 5x = 2\sin(7x)\cdot\cos(2x)$$

2. To express $\cos 4y-\cos 3y$ as a product, the difference to product formula for cosine must be used.

$$\cos\alpha - \cos\beta = -2\sin\left(\frac{\alpha+\beta}{2}\right)\cdot\sin\left(\frac{\alpha-\beta}{2}\right) \to \alpha = 4y$$
$$\to \beta = 3y$$
$$\cos 4y - \cos 3y = -2\sin\left(\frac{4y+3y}{2}\right)\cdot\sin\left(\frac{4y-3y}{2}\right) \to \text{simplify}$$
$$\cos 4y - \cos 3y = -2\sin\left(\frac{7y}{2}\right)\cdot\sin\left(\frac{y}{2}\right)$$

3. To verify $\frac{\cos 3a - \cos 5a}{sin3a + \sin 5a} = -\tan(-a)$, the difference to product formula for cosine and the sum to product formula for sine must be used. In addition, the quotient identity for tangent must be applied.

$$\cos\alpha - \cos\beta = -2\sin\left(\frac{\alpha+\beta}{2}\right)\cdot\sin\left(\frac{\alpha+\beta}{2}\right) \to \alpha = 3a$$
$$\to \beta = 5a$$
$$\cos 3a - \cos 5a = -2\sin\left(\frac{3a+5a}{2}\right)\cdot\sin\left(\frac{3a+5a}{2}\right) \to \text{simplify}$$
$$\cos 3a - \cos 5a = -2\sin 4a \cdot \sin(-a)$$
$$\sin\alpha - \sin\beta = 2\sin\left(\frac{\alpha+\beta}{2}\right)\cdot\cos\left(\frac{\alpha+\beta}{2}\right) \to \alpha = 3a$$
$$\sin 3a - \sin 5a = 2\sin\left(\frac{3a+5a}{2}\right)\cdot\cos\left(\frac{3a+5a}{2}\right) \to \text{simplify}$$
$$\sin 3a - \sin 5a = 2\sin 4a \cdot \cos(-a)$$
$$\frac{\cos 3a - \cos 5a}{\sin 3a + \sin 5a} = -\tan(-a) \to \text{substitute above solutions}$$
$$\frac{-2\sin 4a \cdot \sin(-a)}{2\sin 4a \cdot \cos(-a)} = -\tan(-a) \to \text{simplify}$$
$$\frac{\cancel{-2\sin 4a} \cdot \sin(-a)}{\cancel{2\sin 4a} \cdot \cos(-a)} = -\tan(-a) \to \text{simplify}$$
$$-\frac{\sin(-a)}{\cos(-a)} = -\tan(-a) \to \tan\theta = \frac{\sin\theta}{\cos\theta} \to \theta = -a$$
$$-\frac{\sin(-a)}{\cos(-a)} = -\frac{\sin(-a)}{\cos(-a)}$$
$$-\frac{\sin(-a)}{\cos(-a)} = -\tan(-a)$$

4. To express the product $\sin(6\theta)\sin(4\theta)$ as a sum, the product to sum formula for sine must be used.

$$\sin\alpha\sin\beta = \frac{1}{2}[\cos(\alpha-\beta) - \cos(\alpha+\beta)] \to \alpha = 6\theta$$
$$\to \beta = 4\theta$$
$$\sin(6\theta)\sin(4\theta) = \frac{1}{2}[\cos(6\theta-4\theta) - \cos(6\theta+4\theta)] \to \text{simplify}$$
$$\sin(6\theta)\sin(4\theta) = \frac{1}{2}[\cos(2\theta) - \cos(10\theta)]$$

5. a) To express $5\cos x - 5\sin x$ as a linear combination the formula $a\cos x + b\sin x = C\cos(x-d)$ must be used. From the above, $a = 5$ and $b = -5$. This indicates that the angle is located in the 4th quadrant. The Pythagorean Theorem can be used to determine the value of C .

$$\begin{aligned}
a^2+b^2 &= c^2\\
(5)^2+(-5)^2 &= c^2 \rightarrow \text{simplify}\\
\sqrt{50} &= \sqrt{c^2} \rightarrow \sqrt{\text{both sides}}\\
\sqrt{25\cdot 2} &= c \rightarrow \text{simplify } (\sqrt{50})\\
5\sqrt{2} &= c\\
\cos d &= \frac{\text{adj}}{\text{hyp}}\\
\cos d &= \frac{5}{5\sqrt{2}} \rightarrow \text{rationalize deno minator}\\
\cos d &= \frac{5}{5\sqrt{2}}\left(\frac{\sqrt{2}}{\sqrt{2}}\right) \rightarrow \text{simplify}\\
\cos d &= \frac{5\sqrt{5}}{5\sqrt{4}} = \frac{5\sqrt{2}}{10} = \frac{\sqrt{2}}{2}
\end{aligned}$$

In the 4th quadrant d has a value of $\frac{7\pi}{4}$ radians (unit circle)

$$\begin{aligned}
a\cos x + b\sin x &= C\cos(x-d)\\
5\cos x - 5\sin x &= 5\sqrt{5}\cos\left(x - \frac{7\pi}{4}\right)
\end{aligned}$$

b) To express $-15\cos 3x - 8\sin 3x$ as a linear combination, the formula $a\cos x + b\sin x = C\cos(x-d)$ must be used. From the above, $a = -15$ and $b = -8$. This indicates that the angle is located in the 3rd quadrant. The Pythagorean Theorem can be used to determine the value of C .

$$\begin{aligned}
a^2+b^2 &= c^2\\
(-15)^2+(-8)^2 &= c^2 \rightarrow \text{simplify}\\
\sqrt{289} &= \sqrt{c^2} \rightarrow \sqrt{\text{both sides}}\\
17 &= c\\
\cos d &= \frac{\text{adj}}{\text{hyp}}\\
\cos d &= -\frac{15}{17}\\
\cos^{-1}(\cos d) &= \cos^{-1}\left(\frac{15}{17}\right)\\
d &\approx 28^\circ
\end{aligned}$$

The angle has already been determined to be in the 4th quadrant. Therefore an angle of 28° in standard position in the this quadrant would have a value of approximately 208° or 3.63 radians .

$$
\begin{aligned}
a\cos x + b\sin x &= C\cos(x-d)\\
-15\cos 3x - 8\sin 3x &= 17\cos(x-208^\circ)\\
-15\cos 3x - 8\sin 3x &= 17\cos(x-3.63 \text{ rad})
\end{aligned}
$$

6. To solve the equation $\sin 4x + \sin 2x = 0$ for all values of x such that $0 \le x < 2\pi$, the sum to product formula for sine must be used.

$$
\begin{aligned}
\sin\alpha + \sin\beta &= 2\sin\left(\frac{\alpha+\beta}{2}\right)\cdot\cos\left(\frac{\alpha-\beta}{2}\right) \rightarrow \alpha = 4x\\
&\qquad\qquad\qquad\qquad\qquad\qquad \rightarrow \beta = 2x\\
\sin 4x + \sin 2x &= 2\sin\left(\frac{4x+2x}{2}\right)\cdot\cos\left(\frac{4x-2x}{2}\right) \rightarrow \text{simplify}\\
\sin 4x + \sin 2x &= 2(\sin 3x\cdot\cos x)\\
2(\sin 3x\cdot\cos x) &= 0 \rightarrow \text{solve}\\
\frac{2(\sin 3x\cdot\cos x)}{2} &= \frac{0}{2}\\
\sin 3x\cdot\cos x &= 0\\
\text{Then } \sin 3x &= 0 \text{ Or } \cos x = 0\\
\sin 3x &= 0
\end{aligned}
$$

The interval $0 \le x < 2\pi$ will be tripled since the equation deals with $\sin 3x$. This will give the results in the interval $0 \le x < 6\pi$

$$3x = 0, \pi, 2\pi, 3\pi, 4\pi, 5\pi$$

To obtain the values of x , each of the above answers must be divided by 3 .

$$
\begin{aligned}
x &= 0, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{3\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}\\
x &= 0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, \frac{5\pi}{3}\\
\cos x &= 0\\
x &= \frac{\pi}{2}, \frac{3\pi}{2}
\end{aligned}
$$

When $\sin 4x + \sin 2x = 0$ is solved for all values of x such that $0 \le x < 2\pi$, the results are:

$$x = 0, \frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, \frac{3\pi}{2}, \frac{5\pi}{3}$$

7. To solve the equation $\cos 4x + \cos 2x = 0$ for all values of x such that $0 \le x < 2\pi$, the sum to product formula for cosine must be used.

$$\begin{aligned}
\cos\alpha + \cos\beta &= 2\cos\left(\frac{\alpha+\beta}{2}\right)\cdot\cos\left(\frac{\alpha-\beta}{2}\right) \rightarrow \alpha = 4x \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad \rightarrow \beta = 2x \\
\cos 4x + \cos 2x &= 2\cos\left(\frac{4x+2x}{2}\right)\cdot\cos\left(\frac{4x-2x}{2}\right) \rightarrow \text{simplify} \\
\cos 4x + \cos 2x &= 2\cos 3x\cdot\cos x \\
2\cos 3x\cdot\cos x &= 0 \rightarrow \text{solve} \\
\frac{2\cos 3x)}{2}\cdot\cos x &= \frac{0}{2} \\
\cos 3x\cdot\cos x &= 0 \\
\text{Then } \cos 3x = 0 &\text{ Or } \cos x = 0 \\
\cos 3x &= 0
\end{aligned}$$

The interval $0 \le x < 2\pi$ will be tripled since the equation deals with $\cos 3x$. This will give the results in the interval $0 \le x < 6\pi$

$$3x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, \frac{9\pi}{2}, \frac{11\pi}{2}$$

To obtain the values of x , each of the above answers must be divided by 3 .

$$\begin{aligned}
x &= \frac{\pi}{6}, \frac{3\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{9\pi}{6}, \frac{11\pi}{6} \\
x &= \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{3\pi}{2}, \frac{11\pi}{6} \\
\cos x &= 0 \\
x &= \frac{\pi}{2}, \frac{3\pi}{2}
\end{aligned}$$

When $\cos 4x + \cos 2x = 0$ is solved for all values of x such that $0 \le x < 2\pi$, the results are:

$$x = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{3\pi}{2}, \frac{11\pi}{6}$$

8. To solve the equation $\sin 5x + \sin x = \sin 3x$ for all values of x such that $0 \le x < 2\pi$, the sum to product formula for sine or the difference to product formula for sine must be used. The formula that is used depends upon how the equation is manipulated. However, the solution will not be affected by the formula.

$$
\begin{aligned}
\sin 5x + \sin x &= \sin 3x \rightarrow \text{set } = 0 \\
\sin 5x - \sin 3x + \sin x &= \sin 3x - \sin 3x \\
\sin 5x - \sin 3x + \sin x &= 0 \rightarrow \text{difference to product} \\
\sin\alpha + \sin\beta &= 2\sin\left(\frac{\alpha-\beta}{2}\right)\cdot\cos\left(\frac{\alpha+\beta}{2}\right) \rightarrow \alpha = 5x \\
&\qquad\qquad\qquad\qquad \rightarrow \beta = 3x \\
\sin 5x - \sin 3x &= 2\sin\left(\frac{5x-3x}{2}\right)\cdot\cos\left(\frac{5x+3x}{2}\right) \rightarrow \text{simplify} \\
\sin 5x - \sin 3x &= 2\sin x\cdot\cos x \\
2\sin x\cdot\cos 4x + \sin x &= 0 \rightarrow \text{common factor} \\
\sin x(2\cos 4x + 1) &= 0 \\
\text{Then } \sin x = 0 &\text{ Or } 2\cos 4x + 1 = 0 \\
\sin 3x &= 0 \\
x &= 0, \pi
\end{aligned}
$$

$$
\begin{aligned}
2\cos 4x + 1 &= 0 \\
2\cos 4x + 1 - 1 &= 0 - 1 \\
2\cos 4x &= -1 \\
\frac{2\cos 4x}{2} &= \frac{-1}{2} \\
\cos 4x &= -\frac{1}{2}
\end{aligned}
$$

The interval $0 \leq x < 2\pi$ will be multiplied by 4 since the equation deals with $\cos 4x$. This will give the results in the interval $0 \leq x < 8\pi$

$$4x = \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{8\pi}{3}, \frac{10\pi}{3}, \frac{14\pi}{3}, \frac{16\pi}{3}, \frac{20\pi}{3}, \frac{22\pi}{3}$$

To obtain the values of x , each of the above answers must be divided by 4 .

$$
\begin{aligned}
x &= \frac{2\pi}{12}, \frac{4\pi}{12}, \frac{8\pi}{12}, \frac{10\pi}{12}, \frac{14\pi}{12}, \frac{16\pi}{12}, \frac{20\pi}{12}, \frac{22\pi}{12} \\
x &= \frac{\pi}{6}, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{4\pi}{3}, \frac{5\pi}{3}, \frac{11\pi}{6}
\end{aligned}
$$

When $\sin 5x + \sin x = \sin 3x$ is solved for all values of x such that $0 \leq x < 2\pi$, the results are:

$$x = 0, \frac{\pi}{6}, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{5\pi}{6}, \pi, \frac{7\pi}{6}, \frac{4\pi}{3}, \frac{5\pi}{3}, \frac{11\pi}{6}$$

9. The sum to product formula for sine will be used to simplify the equation $f(t) = \sin(200t + \pi) + \sin(200t - \pi)$

$$\begin{aligned}
\sin\alpha+\sin\beta &= 2\sin\left(\frac{\alpha+\beta}{2}\right)\cdot\cos\left(\frac{\alpha-\beta}{2}\right)\to\alpha=220t+\pi\\
&\qquad\qquad\qquad\qquad\qquad\to\beta=220t-\pi\\
\sin(220t+\pi)+\sin(220t-\pi) &= 2\sin\left(\frac{(220t+\pi)+(220t-\pi)}{2}\right)\cdot\cos\left(\frac{(220t+\pi)-(220t-\pi)}{2}\right)\to\text{simply}\\
\sin(220t+\pi)+\sin(220t-\pi) &= 2\sin\left(\frac{400t}{2}\right)\cdot\cos\left(\frac{2\pi}{2}\right)\to\text{simply}\\
\sin(220t+\pi)+\sin(220t-\pi) &= 2\sin 200t\cdot\cos\pi\to\cos\pi=-1\\
\sin(220t+\pi)+\sin(220t-\pi) &= 2\sin 200t(-1)\\
\sin(220t+\pi)+\sin(220t-\pi) &= -2\sin 200t
\end{aligned}$$

10. To determine a formula for $\tan 4x$ the sum formula for tangent and the double- angle formula for tangent will be used.

$$\tan 4x = \tan(2x+2x) \to \text{sum formula (tan gent)}$$

$$\tan(a+b) = \frac{\tan a + \tan b}{1 - \tan a \tan b} \to a = 2x$$

$$\to b = 2x$$

$$\tan(2x+2x) = \frac{\tan 2x + \tan 2x}{1 - \tan 2x \tan 2x} \to \text{simplify}$$

$$\tan(2x+2x) = \frac{2\tan 2x}{1-\tan^2 2x} \to \text{double angle formula}$$

$$\tan(2x+2x) = \frac{2\left(\frac{2\tan x}{1-\tan^2 x}\right)}{1-\left(\frac{2\tan x}{1-\tan^2 x}\right)^2} \to \text{simplify}$$

$$\tan(2x+2x) = \frac{\left(\frac{4\tan x}{1-\tan^2 x}\right)}{1-\left(\frac{(2\tan x)}{(1-\tan^2 x)}\right)^2} \to \text{common deno min tor}$$

$$\tan(2x+2x) = \frac{\left(\frac{4\tan x}{1-\tan^2 x}\right)}{1\frac{(1-\tan^2 x)^2}{(1-\tan^2 x)^2} - \frac{4tan^2 x}{(1-\tan^2 x)^2}} \to \text{simplify}$$

$$\tan(2x+2x) = \frac{\left(\frac{4\tan x}{1-\tan^2 x}\right)}{\frac{(1-\tan^2 x)^2 - 4\tan^2 x}{(1-\tan^2 x)^2}} \to \text{simplify}$$

$$\tan(2x+2x) = \left(\frac{4\tan x}{1-\tan^2 x}\right) \div \frac{(1-\tan^2 x)^2 - 4\tan^2 x}{(1-\tan^2 x)^2} \to \text{simplify}$$

$$\tan(2x+2x) = \left(\frac{4\tan x}{1-\tan^2 x}\right) \cdot \frac{(1-\tan^2 x)^2}{(1-\tan^2 x)^2 - 4\tan^2 x} \to \text{simplify}$$

$$\tan(2x+2x) = \left(\frac{4\tan x}{\cancel{1-\tan^2 x}}\right) \cdot \frac{\cancel{(1-\tan^2 x)}(1-\tan^2 x)}{(1-\tan^2 x)^2 - 4\tan^2 x} \to \text{simplify}$$

$$\tan(2x+2x) = \frac{4\tan x(1-\tan^2 x)}{(1-\tan^2 x)^2 - 4\tan^2 x} \to \text{expand}$$

$$\tan(2x+2x) = \frac{4\tan x - 4\tan^3 x}{1 - 2\tan^2 x + \tan^4 x - 4\tan^2 x} \to \text{simplify}$$

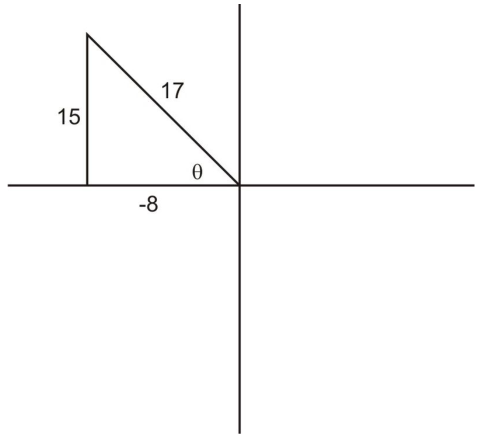
$$\tan(2x+2x) = \frac{4\tan x - 4\tan^3 x}{1 - 6\tan^2 x + \tan^4 x}$$

$$\tan(4x) = \frac{4\tan x - 4\tan^3 x}{1 - 6\tan^2 x + \tan^4 x}$$

Chapter Review

Review Exercises: Pages 280 – 285

1. To determine the sine, cosine and tangent of an angle that has $(-8, 15)$ on its terminal side, sketch the angle in standard position in the 2nd quadrant. Use the Pythagorean Theorem to determine the length of the hypotenuse.

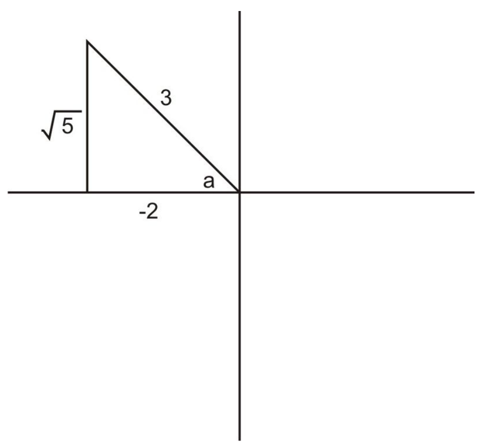


$$\begin{aligned}(h)^2 &= (s_1)^2+(s_2)^2\\(h)^2 &= (15)^2+(-8)^2\\(h)^2 &= 225+64\\\sqrt{h^2} &= \sqrt{289}\\h &= 17\end{aligned}$$

$$\sin\theta = \frac{\text{opp}}{\text{hyp}} \qquad \cos\theta = \frac{\text{adj}}{\text{hyp}} \qquad \tan\theta = \frac{\text{opp}}{\text{adj}}$$

$$\sin\theta = \frac{15}{17} \qquad \cos\theta = -\frac{8}{17} \qquad \tan\theta = -\frac{15}{8}$$

2. If $\sin a = \frac{\sqrt{5}}{3}$ and $\tan a < 0$, the angle in standard position must be located in the 2nd quadrant. Sketch the angle in standard position and use Pythagorean Theorem to determine the length of the adjacent side.



$$\begin{aligned}(h)^2 &= (s_1)^2+(s_2)^2\\(3)^2 &= (\sqrt{5})^2+(s_2)^2\\9 &= 5+(s_2)^2\\\sqrt{4} &= \sqrt{s^2}\\2 &= s\end{aligned}$$

In the second quadrant, this value is negative.

$$\sec a = \frac{\text{hyp}}{\text{adj}}$$
$$\sec a = -\frac{3}{2}$$

3. To simplify $\frac{\cos^4 x - \sin^4 x}{\cos^2 x - \sin^2 x}$, factor both the numerator and denominator using the difference of squares.

$$\frac{\cos^4 x - \sin^4 x}{\cos^2 x - \sin^2 x} \rightarrow \text{factor}$$
$$\frac{(\cos^2 x + \sin^2 x)(\cos^2 x - \sin^2 x)}{(\cos x + \sin x)(\cos x - \sin x)} \rightarrow \text{factor}$$
$$\frac{(\cos^2 x + \sin^2 x)(\cos x + \sin x)(\cos x - \sin x)}{(\cos x + \sin x)(\cos x - \sin x)} \rightarrow \text{simplify}$$
$$\frac{(\cos^2 x + \sin^2 x)\cancel{(\cos x + \sin x)}\cancel{(\cos x - \sin x)}}{\cancel{(\cos x + \sin x)}\cancel{(\cos x - \sin x)}} \rightarrow \text{simplify}$$
$$\cos^2 x + \sin^2 x \rightarrow \text{simplify} \rightarrow \cos^2 x + \sin^2 x = 1$$
$$\frac{\cos^4 x - \sin^4 x}{\cos^2 x - \sin^2 x} = 1$$

4. To verify $\frac{1+\sin x}{\cos x \sin x} = \sec x(\csc x + 1)$, work with one side of the equation, making correct substitutions and performing accurate mathematical computations until both sides read the same.

$$\frac{1+\sin x}{\cos x \sin x} = \sec x(\csc x + 1)$$
$$\frac{1+\sin x}{\cos x \sin x} = \sec x(\csc x + 1) \rightarrow \text{working with LS.}$$
$$\frac{1}{\cos x \sin x} + \frac{\sin x}{\cos x \sin x} = \sec x(\csc x + 1) \rightarrow \text{simplify}$$
$$\left(\frac{1}{\cos x}\right)\left(\frac{1}{\sin x}\right) + \frac{\cancel{\sin x}}{\cos x \cancel{\sin x}} = \sec x(\csc x + 1) \rightarrow \text{simplify}$$
$$\left(\frac{1}{\cos x}\right)\left(\frac{1}{\sin x}\right) + \frac{1}{\cos x} = \sec x(\csc x + 1) \rightarrow \text{reciprocal identities}$$
$$\sec x \cdot \csc x + \sec x = \sec x(\csc x + 1) \rightarrow \text{common factor}$$
$$\sec x(\csc x + 1) = \sec x(\csc x + 1)$$

5. To solve $\sec\left(x + \frac{\pi}{2}\right) + 2 = 0$ for all values of x in the interval $[0, 2\pi)$, the reciprocal identity for secant must be used.

$$\sec\left(x+\frac{\pi}{2}\right)+2=0 \rightarrow \text{solve}$$
$$\sec\left(x+\frac{\pi}{2}\right)+2-2=0-2 \rightarrow \text{simplify}$$
$$\sec\left(x+\frac{\pi}{2}\right)=-2 \rightarrow \sec x=\frac{1}{\cos x}$$
$$\cos\left(x+\frac{\pi}{2}\right)=-\frac{1}{2}$$
$$\cos^{-1}\left(\cos\left(x+\frac{\pi}{2}\right)\right)=\cos^{-1}\left(-\frac{1}{2}\right)$$
$$x+\frac{\pi}{2}=\frac{2\pi}{3},\frac{4\pi}{3} \rightarrow \text{solve for } x$$

$$x+\frac{\pi}{2}=\frac{2\pi}{3}$$
$$x+\frac{\pi}{2}-\frac{\pi}{2}=\frac{2\pi}{3}-\frac{\pi}{2}$$
$$x=\frac{4\pi-3\pi}{6}=\frac{\pi}{6}$$

$$x+\frac{\pi}{2}=\frac{4\pi}{3}$$
$$x+\frac{\pi}{2}-\frac{\pi}{2}=\frac{4\pi}{3}-\frac{\pi}{2}$$
$$x=\frac{8\pi-3\pi}{6}=\frac{5\pi}{6}$$

6. To solve $8\sin\left(\frac{x}{2}\right)-8=0$ for all values of x in the interval $[0,2\pi)$:

$$8\sin\left(\frac{x}{2}\right)-8=0 \rightarrow \text{solve}$$
$$8\sin\left(\frac{x}{2}\right)-8+8=0+8 \rightarrow \text{simplify}$$
$$8\sin\left(\frac{x}{2}\right)=8 \rightarrow \text{simplify}$$
$$\frac{8\sin\left(\frac{x}{2}\right)}{8}=\frac{8}{8} \rightarrow \text{simplify}$$
$$\sin\left(\frac{x}{2}\right)=1 \rightarrow \text{simplify}$$
$$\sin^{-1}\left(\sin\left(\frac{x}{2}\right)\right)=\sin^{-1}(1) \rightarrow \text{simplify}$$
$$\frac{x}{2}=\frac{\pi}{2} \rightarrow \text{solve}$$
$$2x=2\pi$$
$$\frac{2x}{2}=\frac{2\pi}{2}$$
$$x=\pi$$

7. To solve $2\sin^2 x+\sin 2x=0$ for all values of x in the interval $[0,2\pi)$, will involve the double-angle identity for sine and the quotient identity for tangent.

$$2\sin^2 x+\sin 2x=0 \rightarrow \text{double angle identity}$$
$$2\sin^2 x+2\sin x\cos x=0 \rightarrow \text{common factor}$$
$$2\sin x(\sin x+\cos x)=0 \rightarrow \text{solve}$$

$$\text{Then} 2\sin x = 0 \qquad \text{or} \qquad \sin x + \cos x = 0$$

$$\begin{aligned} 2\sin x &= 0 \\ \frac{2\sin x}{2} &= \frac{0}{2} \\ \sin x &= 0 \\ \sin^{-1}(\sin x) &= \sin^{-1}(0) \\ x &= 0, \pi \end{aligned}$$

$$\begin{aligned} &\sin x + \cos x = 0 \to \text{solve} \\ &\sin x + \cos x - \cos x = 0 \to \text{solve} \\ &\sin x = -\cos x \\ &\frac{\sin x}{\cos x} = -\frac{\cos x}{\cos x} \to \text{quotient identity} \\ &\tan x = -1 \\ &\tan^{-1}(\tan x) = \tan^{-1}(-1) \\ &x = -\frac{\pi}{4} \end{aligned}$$

The tangent function is negative in the 2^{nd} and 4^{th} quadrants. Therefore $x = \frac{3\pi}{4}, \frac{7\pi}{4}$

All the values for x are: $x = 0, \frac{3\pi}{4}, \pi, \frac{7\pi}{4}$

8. To solve $3\tan^2 2x = 1$ for all values of x in the interval $[0, 2\pi)$:

$$\begin{aligned} 3\tan^2 2x &= 1 \to \text{solve} \\ \frac{3\tan^2 2x}{3} &= \frac{1}{3} \to \text{simplify} \\ \frac{\cancel{3}\tan^2 2x}{\cancel{3}} &= \frac{1}{3}\text{simplify} \\ \tan^2 2x &= \frac{1}{3} \to \text{simplify} \\ \tan^2 2x &= \frac{1}{3} \to \sqrt{\text{both sides}} \\ \sqrt{\tan^2 2x} &= \sqrt{\frac{1}{3}} \to \text{rationalize denominator} \\ \tan 2x &= \sqrt{\frac{1}{3}\left(\frac{3}{3}\right)}\text{simplify} \\ \tan 2x &= \frac{\sqrt{3}}{3} \to \text{solve} \\ \tan^{-1}(\tan 2x) &= \tan^{-1}\left(\frac{\sqrt{3}}{3}\right) \end{aligned}$$

The interval $0 \le x < 2\pi$ will be doubled since the equation deals with $\tan 2x$. This will give the results in the interval $0 \le x < 4\pi$

$$2x = \frac{\pi}{6}, \frac{7\pi}{6}, \frac{13\pi}{6}, \frac{19\pi}{6}$$

To obtain the values of x , each of the above answers must be divided by 2 .

$$x = \frac{\pi}{12}, \frac{7\pi}{12}, \frac{13\pi}{12}, \frac{19\pi}{12}$$

9. To determine the exact value of $\cos 157.5^\circ$, the half-angle formula for cosine must be used along with the angle 315° .

$$\begin{aligned}
\cos\frac{\theta}{2} &= \pm\sqrt{\frac{\cos\theta+1}{2}} \rightarrow \theta = 315^\circ \\
\cos\frac{315^\circ}{2} &= \pm\sqrt{\frac{\cos 315^\circ+1}{2}} \rightarrow 157.5^\circ(\text{2nd quadrant(-)}) \\
\cos\frac{315^\circ}{2} &= \pm\sqrt{\frac{\frac{\sqrt{2}}{2}+1}{2}} \rightarrow \text{common denominator} \\
\cos\frac{315^\circ}{2} &= \pm\sqrt{\frac{\frac{\sqrt{2}}{2}+\left(\frac{2}{2}\right)1}{2}} \rightarrow \text{simplify} \\
\cos\frac{315^\circ}{2} &= \pm\sqrt{\frac{\frac{\sqrt{2}+2}{2}}{2}} \rightarrow \text{simplify} \\
\cos\frac{315^\circ}{2} &= \pm\sqrt{\left(\frac{\sqrt{2}+2}{2}\right)\left(\frac{1}{2}\right)} \rightarrow \text{simplify} \\
\cos\frac{315^\circ}{2} &= \pm\sqrt{\left(\frac{\sqrt{2}+2}{4}\right)} \rightarrow \text{simplify} \\
\cos\frac{315^\circ}{2} &= \pm\frac{\sqrt{\sqrt{2}+2}}{2}
\end{aligned}$$

10. To determine the exact value of $\frac{13\pi}{12}$, the sine formula for the sum of angles must be used. The angle $\frac{13\pi}{12}$ can be expressed as the sum of $\frac{10\pi}{12}$ and $\frac{13\pi}{12}$.

$$\begin{aligned}
\sin(a+b) &= \sin a\cos b+\cos a\sin b \rightarrow a = \frac{10\pi}{12} \\
&\qquad\qquad\qquad\qquad\qquad \rightarrow b = \frac{3\pi}{12} \\
\sin\left(\frac{10\pi}{12}+\frac{3\pi}{12}\right) &= \sin\left(\frac{10\pi}{12}\right)\cos\left(\frac{3\pi}{12}\right)+\cos\left(\frac{10\pi}{12}\right)\sin\left(\frac{3\pi}{12}\right) \rightarrow \text{simplify} \\
\sin\left(\frac{5\pi}{6}+\frac{\pi}{4}\right) &= \sin\left(\frac{5\pi}{6}\right)\cos\left(\frac{\pi}{4}\right)+\cos\left(\frac{\pi}{6}\right)\sin\left(\frac{\pi}{4}\right) \rightarrow \text{simplify} \\
\sin\left(\frac{5\pi}{6}+\frac{\pi}{4}\right) &= \left(\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right)+\left(-\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) \rightarrow \text{simplify} \\
\sin\left(\frac{5\pi}{6}+\frac{\pi}{4}\right) &= \left(\frac{\sqrt{2}}{4}\right)+\left(-\frac{\sqrt{6}}{4}\right) \rightarrow \text{simplify} \\
\sin\left(\frac{5\pi}{6}+\frac{\pi}{4}\right) &= \frac{\sqrt{4}-\sqrt{6}}{4}
\end{aligned}$$

11. To write $4(\cos 5x+\cos 9x)$ as a product, the sum to product formula for cosine will be used.

$$\begin{aligned}
\cos\alpha+\cos\beta &= 2\cos\left(\frac{\alpha+\beta}{2}\right)\cdot\cos\left(\frac{\alpha-\beta}{2}\right)\to\alpha=5x\\
&\qquad\qquad\qquad\qquad\qquad\qquad\to\beta=9x\\
\cos 5x+\cos 9x &= 2\cos\left(\frac{5x+9x}{2}\right)\cdot\cos\left(\frac{5x-9x}{2}\right)\to\text{simplify}\\
\cos 5x+\cos 9x &= 2\cos(7x)\cdot\cos(-2x)
\end{aligned}$$

12 To simplify $\cos(x-y)\cos y-\sin(x-y)\sin y$, the difference formulas for both cosine and sine must be applied. In addition the Pythagorean Identity $\sin^2 x+\cos^2 x=1$ will be used.

$$\begin{aligned}
&\cos(x-y)=\cos x\cos y+\sin x\sin y\\
&\sin(x-y)=\sin x\cos y-\cos x\sin y\\
&(\cos x\cos y+\sin x\sin y)\cos y-(\sin x\cos y-\cos x\sin y)\sin y\to\text{simplify}\\
&\cos x\cos^2 y+\sin x\sin y\cos y-\sin x\sin y\cos y-\cos x\sin^2 y\to\text{simplify}\\
&\cos x\cos^2 y+\cos x\sin^2 y\to\text{common factor }(\cos x)\\
&\cos x(\cos^2 y+\sin^2 y)\to\sin^2 x+\cos^2 x=1\\
&\cos x(1)\\
&\therefore\cos(x-y)\cos y-\sin(x-y)\sin y=\cos x
\end{aligned}$$

13. To simplify the trigonometric expression $\sin\left(\frac{4\pi}{3}-x\right)+\cos\left(x+\frac{5\pi}{6}\right)$ the difference formula for sine and the sum formula for cosine will both be used.

$$\sin(a-b) = \sin a \cos b - \cos a \sin b \rightarrow a = \frac{4\pi}{3}$$

$$\rightarrow b = x$$

$$\sin\left(\frac{4\pi}{3} - x\right) = \sin\frac{4\pi}{3}\cos x - \cos\frac{4\pi}{3}\sin x$$

$$\cos(a+b) = \cos a \cos b - \sin a \sin b \rightarrow a = x$$

$$\rightarrow b = \frac{5\pi}{6}$$

$$\cos\left(x + \frac{5\pi}{6}\right) = \cos x \cos\frac{5\pi}{6} - \sin x \sin\frac{5\pi}{6}$$

$$\sin\frac{4\pi}{3}\cos x - \cos\frac{4\pi}{3}\sin x + \cos x \cos\frac{5\pi}{6} - \sin x \sin\frac{5\pi}{6} \rightarrow \text{simplify}$$

$$\left(-\frac{\sqrt{3}}{2}\right)\cos x - \left(-\frac{1}{2}\right)\sin x + \cos x\left(-\frac{\sqrt{3}}{2}\right) - \sin x\left(\frac{1}{2}\right) \rightarrow \text{simplify}$$

$$\left(-\frac{\sqrt{3}}{2}\right)\cos x + \frac{1}{2}\sin x - \frac{\sqrt{3}}{2}\cos x - \frac{1}{2}\sin x \rightarrow \text{simplify}$$

$$-2\left(\frac{\sqrt{3}}{2}\right)\cos x \rightarrow \text{simplify}$$

$$\cancel{-2}\left(\frac{\sqrt{3}}{\cancel{2}}\right)\cos x = -\sqrt{3}\cos x$$

$$\sin\left(\frac{4\pi}{3} - x\right) + \cos\left(x + \frac{5\pi}{6}\right) = -\sqrt{3}\cos x$$

14. To derive a formula for $\sin 6x$, the function must be expressed as $\sin\ (4x+2x)$. This means that the sum formula for sine must be used as well as the double angle formula for sine and cosine.

$$\sin(a+b) = \sin a \cos b + \cos a \sin b \rightarrow a = 4x$$

$$\rightarrow b = 2x$$

$$\sin(4x+2x) = \sin 4x \cos 2x + \cos 4x \sin 2x \rightarrow \text{expand}$$

$$\sin(4x+2x) = \sin(2x+2x)\cos 2x + \cos(2x+2x)\sin 2x \rightarrow \text{expand}$$

$$\sin(4x+2x) = \cos 2x(\sin 2x \cos 2x + \cos 2x \sin 2x) + \sin 2x(\cos 2x \cos 2x - \sin 2x \sin 2x) \rightarrow \text{expand}$$

$$\sin(4x+2x) = \sin 2x \cos^2 x + \cos^2 2x \sin 2x + \sin 2x \cos^2 2x - \sin^3 x \rightarrow \text{simplify}$$

$$\sin(4x+2x) = 3\sin 2x \cos^2 x - \sin^3 x \rightarrow \text{common factor}$$

$$\sin(4x+2x) = \sin 2x(3\cos^2 x - \sin^2 x) \rightarrow \text{double angle formula}$$

$$\sin(4x+2x) = 2\sin x \cos x\left[3(\cos^2 x - \sin^2 x)^2 - (2\sin x \cos x)^2\right] \rightarrow \text{simplify}$$

$$\sin(4x+2x) = 2\sin x \cos x\left[3(\cos^4 x - 2\cos^2 x \sin^2 x + \sin^4 x) - 4\sin^2 x \cos^2 x\right] \rightarrow \text{simplify}$$

$$\sin(4x+2x) = 2\sin x \cos x\left[3\cos^4 x - 6\cos^2 x \sin^2 x + 3\sin^4 x - 4\sin^2 x \cos^2 x\right] \rightarrow \text{simplify}$$

$$\sin(4x+2x) = 2\sin x \cos x\left[3\cos^4 x + 3\sin^4 x - 10\sin^2 x \cos^2 x\right] \rightarrow \text{simplify}$$

$$\sin(4x+2x) = 6\sin x \cos^5 x + 6\sin^5 x \cos x - 20\sin^3 x \cos^3 x$$

CHAPTER 4 Inverse Functions and Trigonometric Equations - Solution Key

CHAPTER OUTLINE

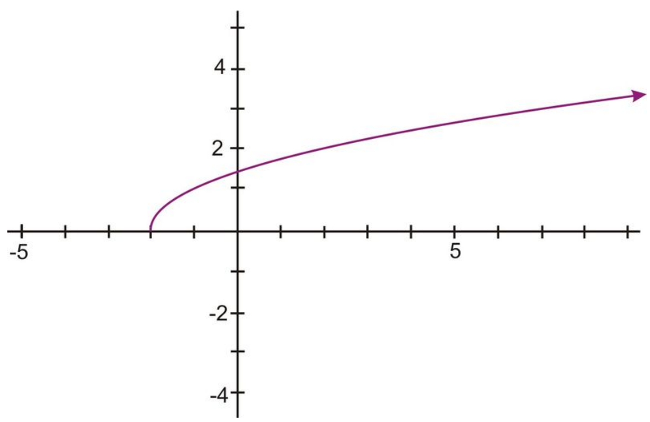
4.1 Inverse Functions and Trigonometric Equations

General Definitions of Inverse Trigonometric Functions

Review Exercises

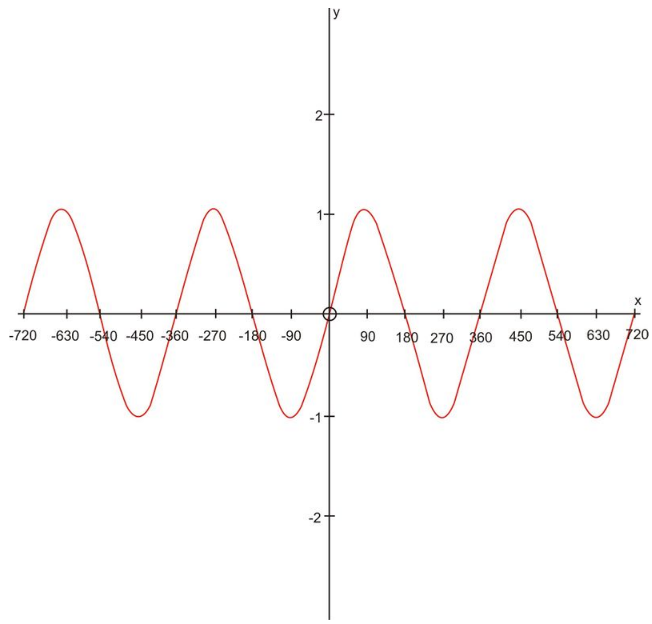
1.

a)



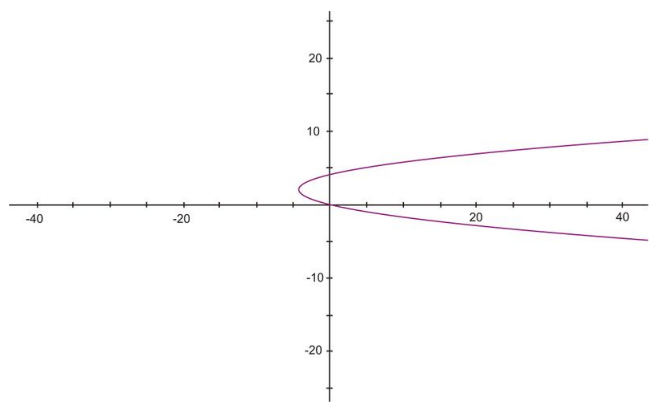
This graph represents a one-to-one function because a vertical line would cross the graph at only one point and a horizontal line would also cross the graph at only one point. Therefore the graph passes both the vertical line test and the horizontal line test. At this point students do know whether or not the function has an inverse that is a function. As a result, it is fine to accept whatever answer the students present as long as they justify their answer.

b)



This graph represents a function because it passes the vertical line test. However, the graph does not pass the horizontal line test. It does not have an inverse that is a function.

c)



The above graph passes the horizontal line test only. It fails the vertical line test. Therefore, this graph does not represent a one-to-one function. It does however, have an inverse that is a function.

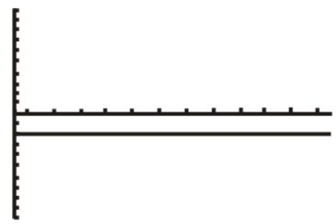
2. To calculate the measure of the angle that the ladder makes with the floor, the trigonometric ratio for cosine must be used. The ladder is the hypotenuse of the right triangle and the distance from the wall is the adjacent side with respect to the reference angle.

$$\begin{aligned}
\cos\ \theta &= \frac{\text{adj}}{\text{hyp}}\\
\cos\ \theta &= \frac{4}{9}\\
\cos\ \theta &= 0.4444\\
\cos^{-1}(\cos^{-1}\ \theta) &= \cos^{-1} = (0.4444)\\
\theta &\approx 63.6^\circ
\end{aligned}$$

1. $\sin^{-1}\left(\frac{\pi}{2}\right)$ does not exist. If π is considered as having an approximate value of 3.14 , then $\frac{3.14}{2} \approx 1.57$. The domain of the sine function is $[-1,1]$.

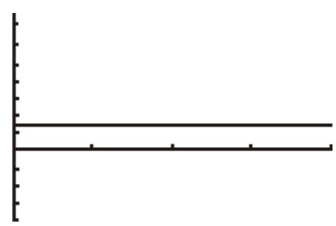
2. $\tan^{-1}(-1)$ does exist. The graph of $\tan^{-1}(-1)$ can be done on the graphing calculator. The exact value is $-\frac{\pi}{4}$.

$y = \tan^{-1}$



3. $\cos^{-1}\left(\frac{1}{2}\right)$ does exist. The graph of $\cos^{-1}\left(\frac{1}{2}\right)$ can be done on the graphing calculator. The exact value is $\frac{\pi}{3}$.

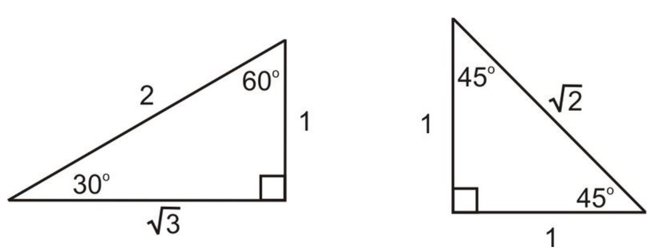
$y = \cos^{-1}\left(\frac{1}{2}\right)$



Ranges of Inverse Circular Functions

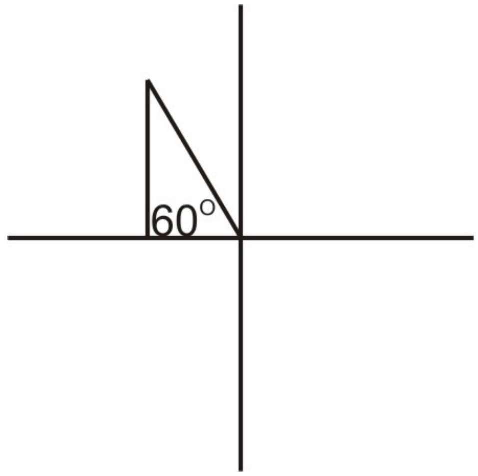
Review Exercises

To determine the exact values of the following functions, the special triangles may be used or the unit circle. The special triangles may be easier for students to sketch and the answers can be readily converted to radians or degrees if necessary.

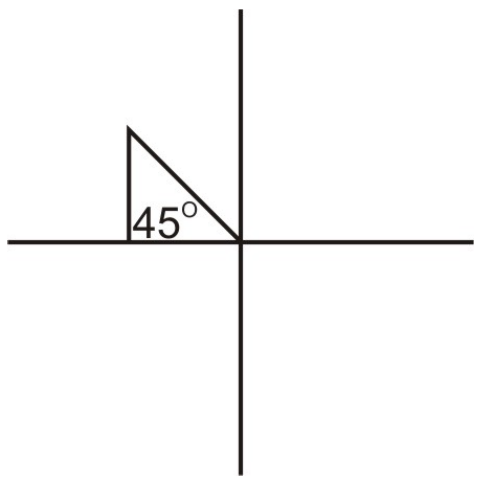


1.

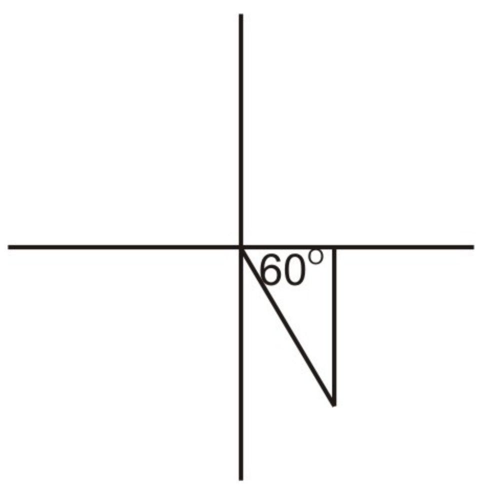
a) $\cos\ 120^\circ$ An angle of 120° has a related angle of 60° in the 2^{nd} quadrant. The cosine function is negative in this quadrant. Using the special triangle, the exact value of $\cos\ 120^\circ$ is $\frac{\text{adj}}{\text{hyp}} = -\frac{1}{2}$



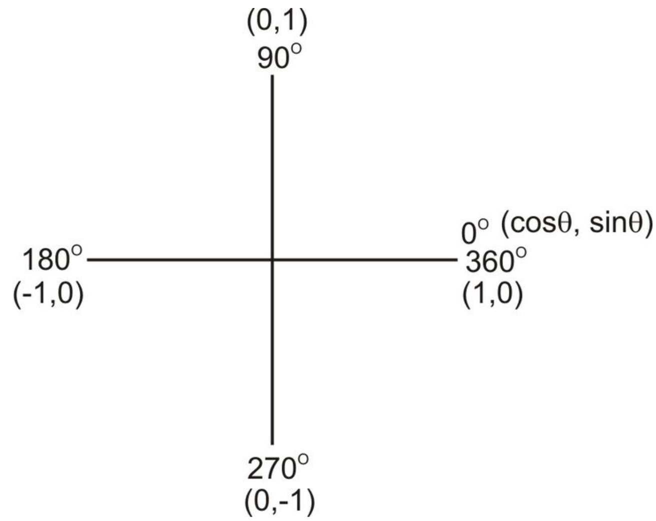
b) $\csc \frac{3\pi}{4}$. An angle of $\frac{3\pi}{4}$rad (135°) has a related angle of $\frac{\pi}{4}$rad (45°) in the 2^{nd} quadrant. Cosecant is the reciprocal of the sine function and is positive in the 2^{nd} quadrant. Therefore, using the special triangle, if $\sin \frac{3\pi}{4} = \frac{1}{\sqrt{2}}$ *then* $\csc \frac{3\pi}{4} = \sqrt{2}$.



c) $\tan \frac{5\pi}{3}$. An angle of $\frac{5\pi}{3}$ rad (300°) has a related angle of $\frac{\pi}{3}$ rad (60°) in the 4^{th} quadrant. The tangent function has a negative value in the 4^{th} quadrant. Using the special triangle, the exact value of $\tan \frac{5\pi}{3}$ is $\frac{\text{opp}}{\text{adj}} = -\frac{\sqrt{3}}{1}$.



a)



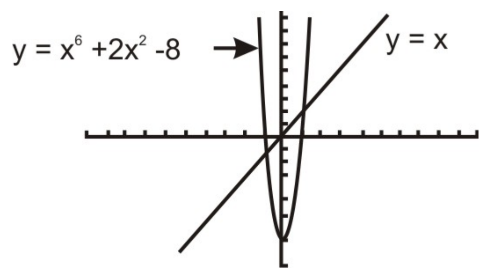
Using this diagram shows that $\cos^{-1}(0) = 90^\circ$ or $\frac{90^\circ}{180^\circ} = \frac{\pi}{2}$rad

b) $\tan^1(-\sqrt{3}) = -60^\circ$ in either the 2$^{\text{nd}}$ quadrant or the 4$^{\text{th}}$ quadrant since the tangent function is negative in these quadrants. The exact value of $\tan^1(-\sqrt{3})$ is $\tan^1(-\sqrt{3}) = -60^\circ$ or $-\frac{\pi}{3}$rad

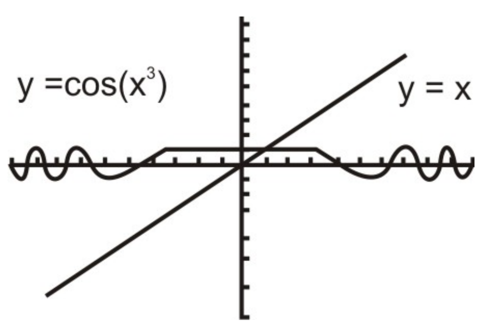
c) $\sin^{-1}\left(-\frac{1}{2}\right) = -30^\circ$ in either the 3$^{\text{rd}}$ or the 4$^{\text{th}}$ quadrant since the sine function is negative in these quadrants. The exact value of $\sin^{-1}\left(-\frac{1}{2}\right) = -30^\circ$ or $-\frac{\pi}{6}$rad is

Review Exercises

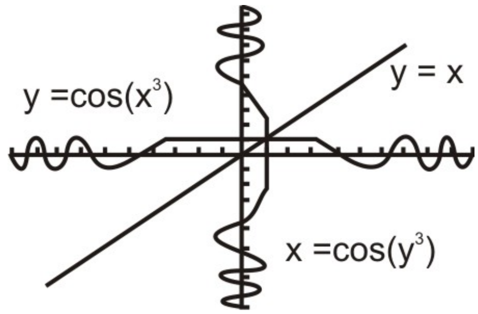
1. The graphs of $y = x^6 + 2x^2 - 8$ and $y = x$ can be graphed using the TI-83. From the graph, it is obvious that the graph of $y = x^6 + 2x^2 - 8$ would not reflect across the line $y = x$ as a mirror image. Therefore the function is not invertible.



b) The graphs of $y = \cos(x^3)$ *and* $y = x$ are shown below as displayed on the TI-83.



The graph of the inverse $x = \cos(y^3)$ is shown below as it appears when added to the above graph on the TI-83.



The function $y = \cos(x^3)$ is invertible because its inverse, $x = \cos(y^3)$, is the mirror image of $y = \cos(x^3)$ reflected across the line $y = x$.

2. To prove that the functions $f(x) = 1 - \frac{1}{x-1}$ and $f^{-1}(x) = 1 + \frac{1}{1-x}$ are inverses, prove algebraically that $f\left(f^{-1}(x)\right) = x$. and $f^{-1}(f(x)) = x$.

$$f(f^{-1}(x)) = 1 - \frac{1}{\left(1 + \frac{1}{1-x}\right) - 1} \rightarrow \text{common denominator}$$

$$f(f^{-1}(x)) = 1 - \frac{1}{\left(1\left(\frac{1-x}{1-x}\right) + \frac{1}{1-x}\right) - 1} \rightarrow \text{simplify}$$

$$f(f^{-1}(x)) = 1 - \frac{1}{\frac{1-x+1}{1-x} - 1} \rightarrow \text{simplify}$$

$$f(f^{-1}(x)) = 1 - \frac{1}{\frac{2-x}{1-x} - \left(\frac{1-x}{1-x}\right)1} \rightarrow \text{simplify} \rightarrow \text{common denominator}$$

$$f(f^{-1}(x)) = 1 - \frac{1}{\frac{2-x-1+x}{1-x}} \rightarrow \text{simplify}$$

$$f(f^{-1}(x)) = 1 - \frac{1}{\frac{1}{1-x}} \rightarrow \text{simplify}$$

$$f(f^{-1}(x)) = 1 - \left[1\left(\frac{1-x}{1}\right)\right] \rightarrow \text{simplify}$$

$$f(f^{-1}(x)) = 1 - (1-x) \rightarrow \text{simplify}$$

$$f(f^{-1}(x)) = 1 - 1 + x \rightarrow \text{simplify}$$

$$f(f^{-1}(x)) = x$$

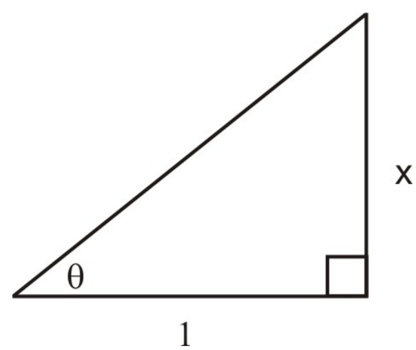
$$f^{-1}(f(x)) = 1 + \frac{1}{1 - \left(1 - \frac{1}{x-1}\right)} \rightarrow \text{common denominator}$$
$$f^{-1}(f(x)) = 1 + \frac{1}{1 - \left(1\left(\frac{x-1}{x-1}\right) - \frac{1}{x-1}\right)} \rightarrow \text{simplify}$$
$$f^{-1}(f(x)) = 1 + \frac{1}{1 - \left(\frac{x-1-1}{x-1}\right)} \rightarrow \text{simplify}$$
$$f^{-1}(f(x)) = 1 + \frac{1}{1\left(\frac{x-1}{x-1}\right) - \left(\frac{x-2}{x-1}\right)} \rightarrow \text{simplify} \rightarrow \text{common denominator}$$
$$f^{-1}(f(x)) = 1 + \frac{1}{\left(\frac{1}{x-1}\right)} \rightarrow \text{simplify}$$
$$f^{-1}(f(x)) = 1 + \left[1\left(\frac{x-1}{1}\right)\right] \rightarrow \text{simplify}$$
$$f^{-1}(f(x)) = 1 + x - 1 \rightarrow \text{simplify}$$
$$f^{-1}(f(x)) = x$$

Derive Properties of Other Five Inverse Circular Functions in Terms of Arctan

Review Exercises

1.

a)



Using this triangle will determine a value for $\tan^{-1}(x)$.

$$\tan\ \theta = \frac{\text{opp}}{\text{adj}}$$
$$\tan\ \theta = \frac{x}{1}$$
$$\tan^{-1}\ (\tan\ \theta) = \tan^{-1}(x)$$
$$\theta = \tan^{-1}(x)$$

$\cos^2(\tan^{-1} x) = \cos^2(\theta)$ Using the same triangle, determine the length of the hypotenuse.

$$\begin{aligned}(h)^2 &= (s_1)^2 + (s_2)^2 \\ (h)^2 &= (x)^2 + (1)^2 \\ (h)^2 &= x^2 + 1 \\ \sqrt{(h)^2} &= \sqrt{x^2+1} \\ h &= \sqrt{x^2+1}\end{aligned}$$

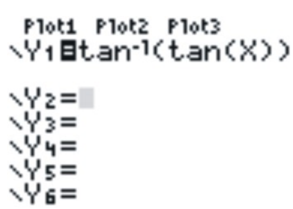
$$\begin{aligned}\cos\ \theta &= \frac{\text{adj}}{\text{hyp}} \\ \cos\ \theta &= \frac{1}{\sqrt{x^2+1}} \\ \cos^2\ \theta &= \left(\frac{1}{\sqrt{x^2+1}}\right)^2 \\ \cos^2\ \theta &= \frac{1}{x^2+1} \\ \therefore\ \cos^2(\tan^{-1} x) &= \frac{1}{x^2+1}\end{aligned}$$

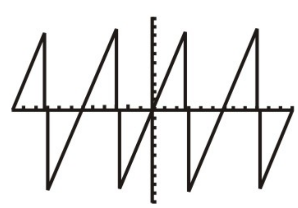
b) $\cot(\tan^{-1} x^2) - \cot^2(\tan^{-1} x)$

As shown above, $\tan^{-1} x = \theta$

$$\begin{aligned}\cot\ \theta &= \frac{\text{adj}}{\text{hyp}} = \frac{1}{x} \\ \cot^2\ \theta &= \left(\frac{1}{x}\right)^2 = \frac{1}{x^2} \\ \therefore\ \cot(\tan^{-1} x^2) &= \frac{1}{x^2}\end{aligned}$$

2. The graph of can be displayed using the TI-83.





The domain is the set of all real numbers except $\frac{\pi}{2} + k\pi$ *where* k is an integer and the range is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Review Exercises

1. To prove $\sin\ \left(\left(\frac{\pi}{2}\right) - \theta\right) = \cos\ \theta$ the cofunction identities for sine and cosine must be used.

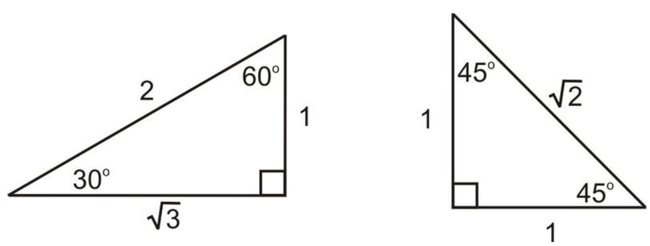
$$\sin\left(\frac{\pi}{2}-\theta\right)=\cos\ \theta \rightarrow \text{cofunction identities}$$
$$\cos\left(\frac{\pi}{2}-\theta\right)=\sin\ \theta$$
$$\sin\left(\left(\frac{\pi}{2}\right)-\theta\right)=\cos\ \left(\frac{\pi}{2}-\left(\frac{\pi}{2}-\theta\right)\right) \rightarrow \text{simplify}$$
$$\sin\left(\left(\frac{\pi}{2}\right)-\theta\right)=\cos\ \left(\frac{\pi}{2}-\frac{\pi}{2}+\theta\right) \rightarrow \text{simplify}$$
$$\sin\left(\left(\frac{\pi}{2}\right)-\theta\right)=\cos(0+\theta)$$
$$\sin\left(\left(\frac{\pi}{2}\right)-\theta\right)=\cos(\theta)$$

2. If $\sin\ \left(\frac{\pi}{2}-\theta\right)=0.68$ and $\sin\ \left(\frac{\pi}{2}-\theta\right)=\cos(\theta)$ *then*

$$-\sin\ \left(\frac{\pi}{2}-\theta\right)=\cos(-\theta)$$
$$\therefore\ \cos(-\theta)=-0.68$$

Review Exercises

1. To determine the exact values of the following inverse functions, the special triangles can be used.

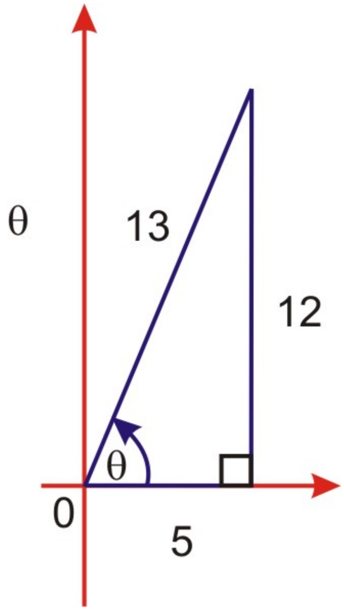


a) $\cos^{-1}\left(\sqrt{\frac{3}{2}}\right)$ From the triangles, it can be verified that $\cos\ \theta(30^\circ)=\frac{\text{adj}}{\text{hyp}}=\frac{\sqrt{3}}{2}$. The exact value of $\cos^{-1}\left(\sqrt{\frac{3}{2}}\right)$ is $\frac{\pi}{6}$.

b) $\sec^{-1}(\sqrt{2})$. The secant function is the reciprocal of the cosine function. Therefore, $\sec\ \theta(45^\circ)=\frac{\text{hyp}}{\text{adj}}=\frac{\sqrt{2}}{1}$. The exact value of $\sec^{-1}(\sqrt{2})$ is $\frac{\pi}{4}$.

c) $\sec^{-1}(-\sqrt{2})$. The secant function is the reciprocal of the cosine function and is therefore negative in the 2$^{\text{nd}}$ and 3$^{\text{rd}}$ quadrants. An angle of 45° in standard position in the 2nd quadrant is an angle of 225° . $\sec\theta(225^\circ)=\frac{\text{hyp}}{\text{adj}}=-\frac{\sqrt{2}}{1}$ The exact value of $\sec^{-1}(\sqrt{2})$ is $\frac{5\pi}{4}$.

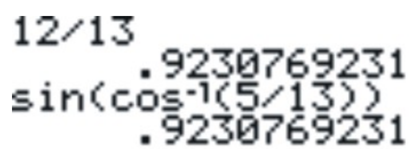
Review Exercises



1. To evaluate $\sin\left(\cos^{-1}\left(\frac{5}{13}\right)\right)$, the angle is located in the 1st quadrant. Working backwards, the previous line to $\cos^{-1}\left(\frac{5}{13}\right)$ is $\cos^{-1}(\cos\ \theta) = \cos^{-1}\left(\frac{5}{13}\right)$. Thus, $\cos\ \theta = \frac{5}{13}$.

$$\sin\left(\cos^{-1}\left(\frac{5}{13}\right)\right) = \sin\ \theta$$
$$\sin\ \theta = \frac{12}{13}.$$

This solution can be verified using technology:



Revisiting

Revisiting $y = c + a\ \cos\ b(x-d)$

Review Exercises

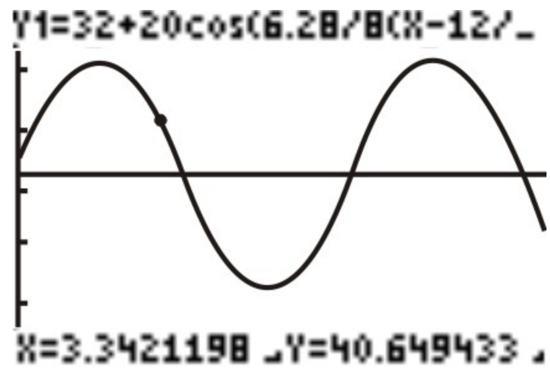
1. The transformations of $y = \cos\ x$ are the vertical reflection; vertical stretch; vertical translation; horizontal stretch and horizontal translation. These changes can be used to write the equation to model a graph of a sinusoidal curve. The simplest way to present these transformations is show them in a list.

$$V.R. = \text{No} \qquad V.S. = \frac{5 - -1}{2} = 3 \qquad V.T. = 2 \qquad H.S. = \frac{210^\circ - 30^\circ}{360^\circ} = \frac{1}{2} \qquad H.T. = 30^\circ$$

The equation that would model the graph of $y = \cos\ x$ that has undergone these transformations is $y = 3\ \cos(2(x - 30^\circ)) + 2$

Review Exercises

1. This problem is an example of an application of solving the equation $y = c + a\cos b(x - d)$ in terms of x . The problem that is presented should be sketched as a graph to facilitate obtaining an equation to model the curve. Once this has been done, the equation can then be entered into the TI-83 and the trace function can be used to estimate a value for x . The following graph was done on the calculator and it shows an estimate of 3.34 seconds for x .



$$y = 32 + \cos\frac{6.28}{8}\left(x - \frac{12}{6.28}\right) \rightarrow \text{equation}$$

$$y = 32 + \cos\frac{6.28}{8}\left(x - \frac{12}{6.28}\right) \rightarrow \text{solve for } x$$

$$x = \frac{\cos^{-1}\left[\frac{y-c}{a}\right]}{b} + d \rightarrow y = 40, c = 32, b = \frac{6.28}{8}, d = \frac{12}{6.28}$$

$$x = \frac{\cos^{-1}\left[\frac{40-32}{20}\right]}{\frac{6.28}{8}} + \frac{12}{6.28}\text{simplify}$$

$$x = \frac{\cos^{-1}\left[\frac{8}{20}\right]}{\frac{6.28}{8}} + \frac{12}{6.28} \rightarrow \text{simplify } u \text{ sin } g\ TI - 83$$

$$x \approx 3.39 \text{ seconds}$$

Solving Trigonometric Equations Analytically

Review Exercises

1. To solve the equation $\sin\ 2\theta = 0.6\, for\ 0 \leq \theta < 2\pi$, involves determining all the possible values for $\sin\ 2\theta = 0.6$ $for\ 0 \leq \theta < 4\pi$ and then dividing these values by 2 to obtain the values for π . The angle is measured in radians since the domain is given in these units.

$$\begin{aligned} \sin\ 2\theta &= 0.6 \rightarrow \text{determine reference angle.} \\ \alpha &= \sin^{-1}(0.6) \\ \alpha &= 0.6435 \end{aligned}$$

The angles for 2θ will be in quadrants $1, 2, 5, 6$.

$$\begin{aligned} 2\theta &= 0.6435, \pi - 0.6435, 2\pi + 0.6435, 3\pi - 0.6435 \\ 2\theta &= 0.6435, 2.4980, 6.9266, 8.7812 \end{aligned}$$

The angles for θ in the domain $[0, 2\pi)$ are:

$$\theta = 0.3218, 1.2490, 3.4633, 4.3906$$

It is not necessary, but these results can be confirmed by using the TI-83 calculator to graph the function.

2. To solve the equation $\cos^2\ x = \frac{1}{16}$ over the interval $[0, 2\pi)$ involves applying the fact that the square root of a number can be positive or negative. This will allow the equation to be solved for all possible values.

$$\begin{aligned} \cos^2\ x &= \frac{1}{16} \rightarrow \sqrt{\text{Both sides}} \\ \sqrt{\cos^2\ x} &= \sqrt{\frac{1}{16}} \rightarrow \text{simplify} \\ \cos\ x &= \pm\frac{1}{4} \\ \cos^{-1}(\cos x) &= \cos^{-1}\left(\frac{1}{4}\right) \end{aligned}$$

Then

$$\begin{aligned} x &= 1.3181 \text{radians} \rightarrow \text{1st eqadrant} \\ x &= 2\pi - 1.3181 \\ x &= 4.9651 \text{radians} \rightarrow \text{4th eqadrant} \\ \cos^{-1}(\cos x) &= \cos^{-1}\left(-\frac{1}{4}\right) \end{aligned}$$

Or

$$\begin{aligned} x &= 1.8235 \text{radians} \rightarrow \text{1st eqadrant} \\ x &= 2\pi - 1.8235 \rightarrow \text{3rd eqadrant} \\ x &= 4.4597 \text{radians} \end{aligned}$$

Once again, the results can be confirmed by graphing the function using the TI-83.

3. To solve the equation $\sin 4\theta - \cos 2\theta = 0$ for all values of θ such that $0 \le \theta \le 2\pi$ involves using the double angle identity for sine.

$$\begin{aligned}
\sin 4\theta - \cos 2\theta &= 0 \\
2\sin 2\theta \cos 2\theta - \cos 2\theta &= 0 \rightarrow \text{common factor} \\
\cos 2\theta(2 \sin 2\theta - 1) &= 0 \rightarrow \text{simplify}
\end{aligned}$$

Then $\cos 2\theta = 0$ over the interval $[0, 4\pi]$

$$\begin{aligned}
2\theta &= \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, \rightarrow \div 2 \\
\theta &= \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}
\end{aligned}$$

Or

$$\begin{aligned}
2 \sin 2\theta - 1 &= 0 \\
2 \sin 2\theta &= 1 \\
\sin 2\theta &= \frac{1}{2} \rightarrow \text{over the interval } [0, 4\pi] \\
2\theta &= \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6} \rightarrow \div 2 \\
\theta & \frac{\pi}{12}, \frac{5\pi}{12}, \frac{13\pi}{12}, \frac{17\pi}{12}
\end{aligned}$$

Once again, the results can be confirmed by graphing the function using the TI-83.

4. To solve the equation $\tan 2x - \cot 2x = 0$ over the interval $0^\circ \le x < 360^\circ$ will involve using the reciprocal identity for cotangent and applying the fact that the square root of a number can be positive or negative. This will allow the equation to be solved for all possible values.

$$\begin{aligned}
\tan 2x - \cot 2x &= 0 \\
\tan 2x - \cot 2x &= 0 \rightarrow \cot x = \frac{1}{\tan x} \\
\tan 2x - \frac{1}{\tan 2x} &= 0 \rightarrow \text{simplify}
\end{aligned}$$

$$\begin{aligned}
\tan 2x(\tan 2x) - (\tan 2x)\frac{1}{\tan 2x} &= (\tan 2x)0 \rightarrow \text{simplify} \\
\tan 2x(\tan 2x) - \cancel{(\tan 2x)}\frac{1}{\cancel{\tan 2x}} &= (\tan 2x)0 \rightarrow \text{simplify} \\
\tan^2 2x - 1 &= 0 \rightarrow \text{simplify} \\
\tan^2 2x &= 1 \rightarrow \sqrt{\text{Both sides}} \\
\sqrt{\tan^2 2x} &= \sqrt{1} \\
\tan 2x &= \pm 1
\end{aligned}$$

Then $\tan\ 2x = 1$ over the interval $[0°, 720°)$. The tangent function is positive in the $1^{\text{st}}, 3^{\text{rd}}, 5^{\text{th}}$ and 7th quadrants.

$$\begin{aligned} 2x &= 45°, 225°, 405°, 5825° \to \div 2 \\ x &= 22.5°, 112.5°, 202.5°, 292.5° \end{aligned}$$

Or $\tan\ 2x = -1$ over the interval $[0°, 720°)$. The tangent function is negative in the $2^{\text{nd}}, 4^{\text{th}}$, 6th , and 8th quadrants.

$$\begin{aligned} 2x &= 135°, 315°, 495°, 675° \\ x &= 67.5°, 157.5°, 247.5°, 337.5° \end{aligned}$$

Once again, the results can be confirmed by graphing the function using the TI-83.

Review Exercises

1. To solve $\sin^2\ x - 2\sin\ x - 3 = 0$ for the values of x that are within the domain of the sine function, involves factoring the quadratic equation and determining the values that fall within the domain of $[0, 2\pi]$ *or* $[0, 360°]$.

$$\begin{aligned} \sin^2\ x - 2\sin\ x - 3 &= 0 \\ \sin^2\ x - 2\sin\ x - 3 &= 0 \to \text{factor} \\ (\sin\ x + 1)(\sin\ x - 3) &= 0 \to \text{simplify} \end{aligned}$$

Then

$$\begin{aligned} \sin\ x + 1 &= 0 \\ \sin\ x &= -1 \\ \sin^{-1}(\sin\ x) &= \sin^{-1}(-1) \\ x &= 270° \text{ or } \frac{3\pi}{2} \end{aligned}$$

Or

$$\begin{aligned} \sin\ x - 3 &= 0 \\ \sin\ x &= 3 \end{aligned}$$

Does not exist. It is not in the range $[-1, 1]$ of the sine function.

2. To solve the equation $\tan^2\ x = 3\tan\ x$ for the principal values of x involves factoring the quadratic equation and determining the values that fall within the domain of the function.

$$\begin{aligned} \tan^2\ x &= 3\ \tan\ x \\ \tan^2\ x - 3\ \tan\ x &= 0 \to \text{common factor} \\ \tan\ x(\tan\ x - 3) &= 0 \to \text{simplify} \end{aligned}$$

Then

$\tan\ x = 0$

$\tan\ x(\tan\ x) = \tan^{-1}(0)$

$x = 0^\circ$

Or

$\tan\ x - 3 = 0$

$\tan\ x = 3$

$\tan^{-1}(\tan\ x) = \tan^{-1}(3)$

$x = 71.5^\circ$

3. To solve the equation $\sin\ x = \cos\frac{x}{2}$ over the interval $[0^\circ, 360^\circ)$ requires the use of the Pythagorean Identity $\sin^2\ \theta + \cos^2\ \theta = 1$ and the half-angle identity for cosine.

$$
\begin{aligned}
\sin\ x &= \cos\frac{x}{2} \\
\sin\ x &= \cos\frac{x}{2} \rightarrow \pm\sqrt{\frac{\cos\ x+1}{2}} \\
\sin\ x &= \pm\sqrt{\frac{\cos\ x+1}{2}} \rightarrow \text{squre both sides} \\
(\sin\ x)^2 &= \left(\pm\sqrt{\frac{\cos\ x+1}{2}}\right)^2 \rightarrow \text{squre both sides} \\
\sin^2\ x &= \frac{\cos x+1}{2} \rightarrow \sin^2\ x+\cos^2\ x = 1 \\
1-\cos^2\ x &= \frac{\cos x+1}{2} \rightarrow \text{simplify} \\
2(1-\cos^2\ x) &= 2\left(\frac{\cos x+1}{2}\right) \rightarrow \text{simplify} \\
2(1-\cos^2\ x) &= \cancel{2}\left(\frac{\cos x+1}{\cancel{2}}\right) \rightarrow \text{simplify} \\
2-2\cos^2\ x &= \cos x+1 \rightarrow \text{simplify}
\end{aligned}
$$

$$2-2\cos^2\ x-\cos x-1 = 0 \rightarrow \text{simplify}$$

$$
\begin{aligned}
-2\cos^2\ x-\cos x+1 &= 0 \rightarrow \div(-1) \\
2\cos^2\ x+\cos x-1 &= 0 \rightarrow \text{factor} \\
(2\ \cos x-1)(\cos x+1) &= 0
\end{aligned}
$$

Then

$2\ \cos x - 1 = 0$

$\cos x = \frac{1}{2}$

$\cos^{-1}(\cos x) = \cos^{-1}\left(\frac{1}{2}\right)$

Cosine is positive in the 1st and 4th quadrants.

$x = 60^\circ, 300^\circ$

Or

$\cos x + 1 = 0$

$\cos x = -1$

$\cos^{-1}(\cos x) = \cos^{-1}(-1)$

Cosine is negative in the 2nd and 3rd quadrant.

$x = 180^\circ$

4. To solve the equation $3 - 3\ \sin^2\ x = 8\sin\ x$ over the interval $[0, 2\pi]$ requires factoring the quadratic equation and solving for all the solutions.

$$
\begin{aligned}
3-3\sin^2 x &= 8\sin x \\
3-3\sin^2 x - 8\sin x &= 8\sin x - 8\sin x \rightarrow \text{simplify} \\
-3\sin^2 x - 8\sin x + 3 &= 0 \rightarrow \div(-1) \\
3\sin^2 x + 8\sin x - 3 &= 0 \rightarrow \text{factor} \\
(3\sin x - 1)(\sin x + 3) &= 0
\end{aligned}
$$

Then

$3\sin x - 1 = 0$

$\sin x = \frac{1}{3}$

$\sin^{-1}(\sin x) = \sin^{-1}\left(\frac{1}{3}\right)$

Sine is positive in the 1st and 2nd quadrants.

$x = 0.3398\text{radians}$

$x = \pi - 0.3398$

$x = 2.8018\text{radians}$

Or

$\sin x + 3 = 0$

$\sin x = -3$

$\sin^{-1}(\sin x)\sin^{-1}(-3)$

Does not exist.

Review Exercises

1. To solve the equation $2\sin x \tan x = \tan x + \sec x$ for all values of $x\varepsilon[0, 2\pi]$ requires the use of the quotient identity for tangent and the reciprocal identity for secant.

$$
\begin{aligned}
2\sin x \tan x &= \tan x + \sec x \\
2\sin x \tan x &= \tan x + \sec x \rightarrow \tan x = \frac{\sin x}{cosx}; \sec x = \frac{1}{\cos x} \\
2\sin x\left(\frac{\sin x}{cosx}\right) &= \left(\frac{\sin x}{cosx}\right) + \left(\frac{1}{cosx}\right) \rightarrow \text{simplify} \\
2\frac{\sin^2 x}{cosx} &= \frac{\sin x + 1}{cosx} \rightarrow \text{simplify} \\
2\left(\frac{\sin^2 x}{\cos x}\right)(\cos x) &= \left(\frac{\sin x + 1}{\cos x}\right)(\cos x) \rightarrow \text{simplify} \\
2\left(\frac{\sin^2 x}{\cancel{\cos x}}\right)\cancel{(\cos x)} &= \left(\frac{\sin x + 1}{\cancel{\cos x}}\right)\cancel{(\cos x)} \rightarrow \text{simplify} \\
2\sin^2 x &= \sin x + 1 \rightarrow \text{simplify} \\
2\sin^2 x - \sin x - 1 &= 0 \rightarrow \text{factor} \\
(2\sin x + 1)(\sin x - 1) &= 0
\end{aligned}
$$

Then

$2 \sin x + 1 = 0$

$\sin x = -\frac{1}{2}$

$\sin^{-1}(\sin x) = \sin^{-1}\left(-\frac{1}{2}\right)$

Sine is negative in the 3rd and 4th quadrants.

$x = \frac{7\pi}{6} \text{and} \frac{11\pi}{6} \text{radians}$

Or

$\sin x - 1 = 0$

$\sin x = 1$

$\sin^{-1}(\sin x) \sin^{-1}(1)$

$x = \frac{\pi}{2} \text{radians}$

2. To solve the equation $\cos 2x = -1 + \cos^2 x$ for all values of x can be simply solved by using the double angle formula for cosine.

$$\begin{aligned}
\cos 2x &= -1 + \cos^2 x \\
\cos 2x &= -1 + \cos^2 x \to \cos(2x) = 2 \cos^2 x - 1 \\
2 \cos^2 x - 1 &= -1 + \cos^2 x \to \text{simplify} \\
2 \cos^2 x - 1 + 1 - \cos^2 x &= 0 \to \text{simplify} \\
\cos^2 x &= 0 \to \sqrt{\text{Both sides}} \\
\sqrt{\cos^2 x} &= \sqrt{0} \\
\cos x &= 0 \\
\cos^{-1}(\cos x) &= \cos^{-1}(0)
\end{aligned}$$

$x = \frac{\pi}{2}$ and for all values of x , $x = \frac{\pi}{2} + k\pi$, *where* $k\varepsilon I$.

3. To solve the equation $2 \cos^2 x + 3 \sin x - 3 = 0$ for all values of x over the interval $[0, 2\pi]$ requires the use of the Pythagorean Identity $\sin^2 \theta + \cos^2 \theta = 1$.

$$\begin{aligned}
2\cos^2 x + 3\sin x - 3 &= 0 \\
2\cos^2 x + 3\sin x - 3 &= 0 \to \sin^2 x + \cos^2 x = 1 \\
2(1 - \sin^2 x) + 3\sin x - 3 &= 0 \to \text{expand} \\
2 - 2\sin^2 x + 3\sin x - 3 &= 0 \to \text{simlify} \\
-2\sin^2 x + 3\sin x - 1 &= 0 \to \div(-1) \\
2\sin^2 x - 3\sin x + 1 &= 0 \to \text{factor} \\
(2\sin x - 1)(\sin x - 1) &= 0
\end{aligned}$$

Then

$2\sin x - 1 = 0$

$\sin x = \frac{1}{2}$

$\sin^{-1}(\sin x) = \sin^{-1}\frac{1}{2}$

Sine is positive in the 1st and 2nd quadrants

$x = \frac{\pi}{6}$ and $\frac{5\pi}{6}$ radians

Or

$\sin x - 1 = 0$

$\sin x = 1$

$\sin^{-1}(\sin x) = \sin^{-1}(1)$

$x = \frac{\pi}{2}$

Review Exercises

1. To solve the equation $3\cos^2 x - 5\sin x = 4$ for all values of x over the interval $0^\circ \leq x \leq 360^\circ$ will require writing the equation in terms of sine by using the Pythagorean Identity $\sin^2 \theta + \cos^2 \theta = 1$ and then using the quadratic formula to solve the equation.

$$\begin{aligned}
3\cos^2 x - 5\sin x &= 4\\
3\cos^2 x - 5\sin x &= 4 \to \sin^2 x + \cos^2 x = 1\\
3(1-\sin^2 x) - 5\sin x &= 4 \to \text{expand}\\
3 - 3\sin^2 x - 5\sin x &= 4 \to \text{simplify}\\
3 - 3\sin^2 x - 5\sin x - 4 &= 4 - 4 \to \text{simplify}\\
-3\sin^2 x - 5\sin x - 1 &= 0 \to \div(-1)\\
3\sin^2 x + 5\sin x + 1 &= 0 \to \div(-1) \text{ Let } y = \sin x\\
3y^2 + 5y + 1 &= 0\\
a = 3 \ b &= 5 \ c = 1\\
y &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}\\
y &= \frac{-5 \pm \sqrt{(5)^2 - 4(3)(1)}}{2(3)} \to \text{simplify}\\
y &= \frac{-5 \pm \sqrt{13}}{6} \to \text{simplify}
\end{aligned}$$

Then

$y = \frac{-5 + \sqrt{13}}{6}$

$y \approx -0.2324$

$y = \sin x$

$\sin x = -0.2324$

$\sin^{-1}(\sin x) = \sin^{-1}(-0.2324)$

Sine is negative in the $3^{\text{rd}}$and 4^{th}quadrants

$x \approx 193.5^\circ$ and $x \approx 346.5^\circ$

Or

$y = \frac{-5 - \sqrt{13}}{6}$

$y \approx -1.4342$

$y = \sin x$

$\sin x = -1.4342$

$\sin^{-1}(\sin x) = \sin^{-1}(-1.4342)$

Does not exist.

2. The quadratic formula must be used to solve the trigonometric equation $\tan^2 x + \tan x + 2 = 0$ for values of x over the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$$\begin{aligned}
\tan^2 x + \tan x + 2 &= 0 \\
\tan^2 x + \tan x + 2 &= 0 \text{ Let } y = \tan x \\
y^2 + y + 2 &= 0 \\
a = 1 \; b = 1 \; c &= 2 \\
Y &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\
Y &= \frac{-1 \pm \sqrt{(1)^2 - 4(1)(-2)}}{2(1)} \rightarrow \text{simplify} \\
Y &= \frac{-1 \pm \sqrt{9}}{2(1)} \rightarrow \text{simplify}
\end{aligned}$$

Then	Or
$y = \frac{-1 + \sqrt{9}}{2(1)} \rightarrow \text{simplify}$	$y = \frac{-1 - \sqrt{9}}{2(1)} \rightarrow \text{simplify}$
$y = 1$	$y = -2$
$y = \tan x$	$y = \tan x$
$\tan x = 1$	$\tan x = -2$
$\tan^{-1}(\tan x) = \tan^{-1}(1)$	$\tan^{-1}(\tan x) = \tan^{-1}(-2)$
$x = \frac{\pi}{4} + k\pi$	$x = \arctan(-2) + k\pi$

3. To solve the equation $5\cos^2 \theta - 6\sin \theta = 0$ over the interval $[0, 2\pi]$ involves using the Pythagorean Identity $\sin^2 \theta + \cos^2 \theta = 1$ and the quadratic formula.

$$\begin{aligned}
5\cos^2 \theta - 6\sin \theta &= 0 \\
5\cos^2 \theta - 6\sin \theta &= 0 \rightarrow \sin^2 \theta + \cos^2 \theta = 1 \\
5(1 - \sin^2 \theta) - 6\sin \theta &= 0 \rightarrow \text{expand} \\
5 - 5\sin^2 \theta - 6\sin \theta &= 0 \rightarrow \text{simplify} \\
-5\sin^2 \theta - 6\sin \theta + 5 &= 0 \rightarrow \div(-1) \\
5\sin^2 \theta + 6\sin \theta - 5 &= 0 \rightarrow \text{solve Let } y = \sin x \\
5y^2 + 6y - 5 &= 0 \\
a = 5 \; b = 6 \; c &= -5 \\
y &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\
y &= \frac{-6 \pm \sqrt{(6)^2 - 4(5)(-5)}}{2(5)} \rightarrow \text{simplify} \\
y &= \frac{-6 \pm \sqrt{136}}{10} \rightarrow \text{simplify}
\end{aligned}$$

Then

$$y = \frac{-6+11.66}{10} \to \text{simplify}$$
$$y = 0.566$$
$$y = \sin\ x$$
$$\sin\ x = 0.566$$
$$\sin^{-1}(\sin\ x) = \sin^{-1}(0.566)$$
$$x \approx 0.6016\text{radians} \pm 2\pi$$
$$x \approx 2.5399\text{radians} \pm 2\pi$$

Or

$$y = \frac{-6-11.66}{10} \to \text{simplify}$$
$$y = -1.766$$
$$y = \sin\ x$$
$$\sin\ x = -1.766$$
$$\sin^{-1}(\sin\ x) = \sin^{-1}(1.766)$$

Dose not exit

Solve Equations (with double angles)

Review Exercises

1. If $\tan\ x = \frac{3}{4}$ and $0^\circ < x < 90^\circ$, the angle is in standard position in the 1^{st} quadrant. The triangle is a $3-4-5$ triangle which makes $\sin\ x = \frac{3}{5}$ *and* $\cos\ x = \frac{4}{5}$. The value of $\tan\ 2x$ can be found by using the double angle formula for tangent.

a)

$$\tan\ (2x) = \frac{2\tan\ x}{1-\tan^2\ x}$$
$$\tan\ (2x) = \frac{2\left(\frac{3}{4}\right)}{1-\left(\frac{3}{4}\right)^2} \to \text{simplify}$$
$$\tan\ (2x) = \frac{\cancel{2}\left(\frac{3}{\cancel{4}_2}\right)}{1-\frac{9}{16}} \to \text{common deno min ator}$$
$$\tan\ (2x) = \frac{\frac{3}{2}}{1\left(\frac{16}{16}\right)-\frac{9}{16}} \to \text{simplify}$$
$$\tan\ (2x) = \frac{\frac{3}{2}}{\frac{16-9}{16}} \to \text{simplify}$$
$$\tan\ (2x) = \frac{\frac{3}{2}}{\frac{7}{16}} \to \text{simplify}$$
$$\tan\ (2x) = \frac{3}{2}\cdot\left(\frac{16}{7}\right) \text{simplify}$$
$$\tan\ (2x) = \frac{24}{7} \approx 3.4286$$

b) The value of $\sin\ 2x$ can be found by using the double angle formula for sine and the values of $\sin\ x$ which is $\frac{3}{5}$ and of $\cos\ x$ which is $\frac{4}{5}$.

$$\begin{aligned}
\sin\,(2x) &= 2\sin\,x\,\cos\,x \\
\sin\,(2x) &= 2\sin\,x\,\cos\,x \to \sin\,x = \frac{3}{5} \\
&\qquad\qquad\qquad\quad \to \cos\,x = \frac{4}{5} \\
\sin\,(2x) &= 2\left(\frac{3}{5}\right)\cdot\left(\frac{4}{5}\right) \to \text{simplify} \\
\sin\,(2x) &= \frac{24}{25} = 0.960
\end{aligned}$$

c) The value of $\cos\,2x$ can be found by using the double angle formula for cosine and the values of $\sin\,x$ which is $\frac{3}{5}$ and of which is $\frac{4}{5}$.

$$\begin{aligned}
\cos\,(2x) &= \cos^2\,x\,\sin^2\,x \\
\cos\,(2x) &= \cos^2\,x - \sin^2\,x \to \sin\,x = \frac{3}{5} \\
&\qquad\qquad\qquad\qquad \to \cos\,x = \frac{4}{5} \\
\cos\,(2x) &= \left(\frac{4}{5}\right)^2\cdot\left(\frac{3}{5}\right)^2 \quad \to \text{simplify} \\
\cos\,(2x) &= \frac{16-9}{25} \to \text{simplify} \\
\cos\,(2x) &= \frac{7}{25} = 0.280
\end{aligned}$$

2. To prove that $2\ \csc(2x) = \csc^2\,x\,\tan\,x$ is an identity, work with the left side. The reciprocal identity for sine must be used as well as the double angle formula for sine.

$$\begin{aligned}
2\ \csc(2x) &= \csc^2\ x\ \tan\ x \\
2\ \csc(2x) &= \csc^2\ x\ \tan\ x \rightarrow \csc\ x = \frac{1}{\sin\ x} \\
2\left(\frac{1}{\sin(2x)}\right) &= \csc^2\ x\ \tan\ x \rightarrow \text{simplify} \\
\frac{2}{\sin(2x)} &= \csc^2\ x\ \tan\ x \rightarrow \text{double angle formula} \\
\frac{2}{2\sin\ x\ \cos\ x} &= \csc^2\ x\ \tan\ x \rightarrow \text{simplify} \\
\frac{\cancel{2}}{\cancel{2}\sin\ x\ \cos\ x} &= \csc^2\ x\ \tan\ x \rightarrow \text{simplify} \\
\frac{1}{\sin\ x\ \cos\ x} &= \csc^2\ x\ \tan\ x \rightarrow \text{multiply left side by } \frac{\sin\ x}{\sin\ x} \\
\left(\frac{\sin\ x}{\sin\ x}\right)\frac{1}{\sin\ x\ \cos\ x} &= \csc^2\ x\ \tan\ x \rightarrow \text{simplify} \\
\frac{\sin\ x}{\sin^2\ x\ \cos x} &= \csc^2\ x\ \tan\ x \rightarrow \text{express as factors} \\
\left(\frac{1}{\sin^2\ x}\right)\cdot\left(\frac{\sin\ x}{\cos\ x}\right) &= \csc^2\ x\ \tan\ x \rightarrow \frac{1}{\sin\ x} = \cos\ x \\
& \qquad\qquad\qquad\quad \rightarrow \frac{\sin\ x}{\cos\ x} = \tan\ x \\
\csc^2\ x\ \tan\ x &= \csc^2\ x\ \tan\ x
\end{aligned}$$

b) To prove that $\cos^4\ \theta - \sin^4\ \theta = \cos\ 2\theta$ is an identity, work with the left side. The left side must be factored by using the difference of two squares and then the Pythagorean Identity $\sin^2\ \theta + \cos^2\ \theta = 1$ must be applied.

$$\begin{aligned}
\cos^4\ \theta - \sin^4\ \theta &= \cos\ 2\theta \\
\cos^4\ \theta - \sin^4\ \theta &= \cos\ 2\theta \rightarrow \text{factor} \\
(\cos^2\ \theta - \sin^2\ \theta)(\cos^2\ \theta - \sin^2\ \theta) &= \cos\ 2\theta \rightarrow \sin^2\ \theta + \sin^2\ \theta = 1 \\
1(\cos^2\ \theta - \sin^2\ \theta) &= \cos\ 2\theta \rightarrow \text{simplify} \\
(\cos^2\ \theta - \sin^2\ \theta) &= \cos\ 2\theta \rightarrow \cos^2\ \theta - \sin^2\ \theta = \cos\ 2\theta \\
\cos\ 2\theta &= \cos\ 2\theta
\end{aligned}$$

c) To prove that $\frac{\sin\ 2x}{1+\cos\ 2x} = \tan\ x$ is an identity, work with the left side. The double angle formula for sine and the double angle formula for cosine must be used.

$$
\begin{aligned}
\frac{\sin\ 2x}{1+\cos\ 2x} &= \tan\ x \\
\frac{\sin\ 2x}{1+\cos\ 2x} &= \tan\ x \to \sin\ 2x = 2\ \sin\ x\ \cos x \\
&\qquad\qquad \to \cos\ 2x = 1-2\ \sin^2\ x \\
\frac{2\ \sin\ x\ \cos x}{1+(1-2\ \sin^2\ x)} &= \tan\ x \to \text{simplify} \\
\frac{2\ \sin\ x\ \cos x}{2-2\ \sin^2\ x} &= \tan\ x \to \text{common factor} \\
\frac{2\ \sin\ x\ \cos x}{2(1-\sin^2\ x)} &= \tan\ x \to \sin^2\ x+\cos^2\ x = 1 \\
\frac{2\ \sin\ x\ \cos x}{2\ \cos^2\ x} &= \tan\ x \to \text{factor} \\
\frac{2\ \sin\ x\ \cos x}{2(\cos x)(\cos x)} &= \tan\ x \to \text{simplify} \\
\frac{\cancel{2}\ \sin\ x\ \cancel{\cos x}}{\cancel{2}(\cos x)\cancel{(\cos x)}} &= \tan\ x \to \frac{\sin\ x}{\cos\ x} = \tan\ x \\
\frac{\sin\ x}{\cos\ x} &= \tan\ x \\
\tan\ x &= \tan\ x
\end{aligned}
$$

3. To solve the trigonometric equation $\cos\ 2\theta = 1-2\sin^2\ \theta$ such that $-\pi \leq \theta < \pi$ involves using the double angle formula for cosine.

4. To solve the trigonometric equation $\cos\ 2x = \cos x$ such that $0 \leq x < \pi$ involves using the double angle formula for cosine.

$$
\begin{aligned}
\cos\ 2x &= \cos x \\
\cos\ 2x &= \cos x \to 2\ \cos^2\ x - 1 = \cos\ 2x \\
2\cos^2\ x - 1 &= \cos x \to \text{simplify} \\
2\cos^2\ x - 1 - \cos x &= 0 \to \text{simplify} \\
2\cos^2\ x - \cos x - 1 &= 0 \to \text{factor} \\
(2\ \cos x + 1)(\cos x - 1) &= 0
\end{aligned}
$$

Then	Or
$2\ \cos x + 1 = 0$	$\cos x - 1 = 0$
$\cos x = -\frac{1}{2}$	$\cos x = 1$
$\cos^{-1}(\cos x) = \cos^{-1}\left(-\frac{1}{2}\right)$	$\cos^{-1}(\cos x) = \cos^{-1}(1)$
Cosine is negative in the 2nd quadrant	$= 0$
$x = \frac{2\pi}{3}$radians	

Review Exercises

1. a) To determine the exact value of $\sin 67.5^\circ$, the half-angle identity for sine will be used with an angle of $\frac{135^\circ}{2}$. The special triangles will also be used.

$$\sin\frac{\theta}{2}=\pm\sqrt{\frac{1-\cos\,\theta}{2}}$$
$$\sin\frac{\theta}{2}=\pm\sqrt{\frac{1-\cos\,\theta}{2}}\rightarrow\theta=135^\circ$$
$$\sin\frac{135^\circ}{2}=\pm\sqrt{\frac{1-\cos 135^\circ}{2}}\rightarrow\cos 135^\circ=-\frac{1}{\sqrt{2}}$$
$$\sin\frac{135^\circ}{2}=\pm\sqrt{\frac{1-\left(-\frac{1}{\sqrt{2}}\right)}{2}}\rightarrow\text{simplify}$$
$$\sin\frac{135^\circ}{2}=\pm\sqrt{\frac{1+\frac{1}{2}}{2}}\rightarrow\text{common deno min ator}$$
$$\sin\frac{135^\circ}{2}=\pm\sqrt{\frac{1\left(\frac{\sqrt{2}}{\sqrt{2}}\right)+\frac{1}{\sqrt{2}}}{2}}\rightarrow\text{simplify}$$
$$\sin\frac{135^\circ}{2}=\pm\sqrt{\frac{\frac{\sqrt{2}+1}{\sqrt{2}}}{2}}\rightarrow\text{simplify}$$
$$\sin\frac{135^\circ}{2}=\pm\sqrt{\left(\frac{\sqrt{2}+1}{\sqrt{2}}\right)\cdot\left(\frac{1}{2}\right)}\rightarrow\text{simplify}$$
$$\sin\frac{135^\circ}{2}=\pm\sqrt{\left(\frac{\sqrt{2}+1}{2\sqrt{2}}\right)}\rightarrow\text{rationalize deno min ator}$$
$$\sin\frac{135^\circ}{2}=\pm\sqrt{\left(\frac{\sqrt{2}+1}{2\sqrt{2}}\right)\left(\frac{\sqrt{2}}{\sqrt{2}}\right)}\rightarrow\text{simplify}$$
$$\sin\frac{135^\circ}{2}=\pm\sqrt{\left(\frac{\sqrt{4}+\sqrt{2}}{2\sqrt{4}}\right)}\rightarrow\text{simplify}$$
$$\sin\frac{135^\circ}{2}=\pm\sqrt{\left(\frac{2+\sqrt{2}}{4}\right)}\rightarrow\text{simplify}$$
$$\sin\frac{135^\circ}{2}=\pm\frac{\sqrt{\sqrt{2}+2}}{2}$$

An angle of 67.5° is located in the 1^{st} quadrant and the sine of an angle in this quadrant is positive.

$$\therefore\sin\frac{135^\circ}{2}=\frac{\sqrt{\sqrt{2}+2}}{2}$$

b) To determine the exact value of $\tan 165^\circ$, the half-angle identity for tangent will be used with an angle of $\frac{330^\circ}{2}$. The special triangles will also be used.

$$\tan\frac{\theta}{2} = \frac{1-\cos\ \theta}{\sin\ \theta}$$
$$\tan\frac{\theta}{2} = \frac{1-\cos\ \theta}{\sin\ \theta} \rightarrow \theta = 330^\circ$$
$$\tan\frac{330^\circ}{2} = \frac{1-\cos\ 330^\circ}{\sin\ 330^\circ} \rightarrow \cos\ 330^\circ = \frac{\sqrt{3}}{3}$$
$$\rightarrow \sin\ 330^\circ = -\frac{1}{2}$$
$$\tan\frac{330^\circ}{2} = \frac{1-\frac{\sqrt{3}}{2}}{-\frac{1}{2}} \rightarrow \text{commom denominator}$$
$$\tan\frac{330^\circ}{2} = \frac{1\left(\frac{2}{2}\right)-\frac{\sqrt{3}}{2}}{-\frac{1}{2}} \rightarrow \text{simplify}$$
$$\tan\frac{330^\circ}{2} = \frac{\frac{2-\sqrt{3}}{2}}{-\frac{1}{2}} \rightarrow \text{simplify}$$
$$\tan\frac{330^\circ}{2} = \frac{2-\sqrt{3}}{2}\cdot\left(-\frac{2}{1}\right) \rightarrow \text{simplify}$$
$$\tan\frac{330^\circ}{2} = \frac{-4+2\sqrt{3}}{2} \rightarrow \text{simplify}$$
$$\tan\frac{330^\circ}{2} = \frac{-\cancel{4}2+\cancel{2}\sqrt{3}}{\cancel{2}}$$
$$\tan\frac{330^\circ}{2} = -2+\sqrt{3}$$

2. To prove that $\sin\ x\ \tan\left(\frac{x}{2}\right)+2\ \cos x = 2\ \cos^2\left(\frac{x}{2}\right)$ work with both sides of the equation and use the half-angle identity for cosine and the half-angle identity for tangent.

$$\sin\ x\ \tan\left(\frac{x}{2}\right)+2\ \cos x = 2\ \cos^2\left(\frac{x}{2}\right)$$

Left Side:	Right Side
$\sin\ x\ \tan\left(\frac{x}{2}\right)+2\ \cos x$	$2\ \cos^2\left(\frac{x}{2}\right)$
$\sin\ x\ \tan\left(\frac{x}{2}\right)+2\ \cos x \rightarrow \tan\left(\frac{x}{2}\right) = \frac{1-\cos x}{\sin x}$	$2\ \cos^2\left(\frac{x}{2}\right) \rightarrow \cos\frac{x}{2} = \pm\sqrt{\frac{\cos x+1}{2}}$
$\sin x\left(\frac{1-\cos x}{\sin x}\right)+2\ \cos x \rightarrow \text{simplify}$	$2\left(\pm\sqrt{\frac{\cos x+1}{2}}\right)^2 \rightarrow \text{simplify}$
$\cancel{\sin x}\left(\frac{1-\cos x}{\cancel{\sin x}}\right)+2\ \cos x \rightarrow \text{simplify}$	$2\left(\frac{\cos x+1}{2}\right) \rightarrow \text{simplify}$
$1-\cos x+2\ \cos x \rightarrow \text{simplify}$	$\cancel{2}\left(\frac{\cos x+1}{\cancel{2}}\right)$
$1+\cos x$	$\cos x+1$

Since both sides of the equation equal $1+\cos x$, they are equal to each other.

3. To solve the trigonometric equation $\cos\frac{x}{2} = 1+\cos x$ such that $0 \le x < 2\pi$ the half- angle identity for cosine must be applied.

$$\begin{aligned}
\cos\frac{x}{2} &= 1+\cos x \\
\cos\frac{x}{2} &= 1+\cos x \rightarrow \cos\frac{x}{2} = \pm\sqrt{\frac{\cos x+1}{2}} \\
\pm\sqrt{\frac{\cos x+1}{2}} &= 1+\cos x \rightarrow \text{square both sides} \\
\left(\pm\sqrt{\frac{\cos x+1}{2}}\right)^2 &= (1+\cos x)^2 \rightarrow \text{expand} \\
\frac{\cos x+1}{2} &= 1+2\ \cos x+\cos^2 x \rightarrow \text{simplify} \\
2\left(\frac{\cos x+1}{2}\right) &= 2(1+2\ \cos x+\cos^2 x) \rightarrow \text{simplify} \\
\cancel{2}\left(\frac{\cos x+1}{\cancel{2}}\right) &= 2(1+2\ \cos x+\cos^2 x) \rightarrow \text{simplify} \\
\cos x+1 &= 2+4\ \cos x+2\ \cos^2 x \rightarrow \text{simplify} \\
\cos x-\cos x+1-1 &= 2+4\ \cos x+2\ \cos^2 x-\cos x-1 \rightarrow \text{simplify} \\
0 &= 2\ \cos^2 x+3\cos x+1 \rightarrow \text{simplify} \\
2\ \cos^2 x+3\cos x+1 &= 0 \rightarrow \text{solve} \\
(2\ \cos x+1)(\cos x+1) &= 0
\end{aligned}$$

Then

$2\ \cos x+1 = 0$

$\cos x = -\frac{1}{2}$

$\cos^{-1}(\cos x) = \cos^{-1}\left(-\frac{1}{2}\right)$

The cosine function is negative in the 2nd and 3rd quadrants.

$x = \frac{2\pi}{3}$ and $\frac{4\pi}{3}$ radians

Or

$\cos x+1 = 0$

$\cos x = -1$

$\cos^{-1}(\cos x) = \cos^{-1}(-1)$

$x = \pi$

Review Exercises

1.

$$\begin{aligned}
1-\sin x &= \sqrt{3}\sin x \rightarrow \text{isolate} \sin x\\
1 &= \sqrt{3}\sin x+\sin x \rightarrow \text{simplify}\\
1 &= 2.73\sin x \rightarrow \text{solve}\\
\frac{1}{2.7321} &= \frac{2.7321\ \sin x}{2.7321}\\
0.3660 &= \sin x\\
\sin^{-1}(0.3660) &= \sin^{-1}\sin x\\
0.3747\text{radians} &= x
\end{aligned}$$

Over the interval $[0,\pi]$ the sine function is positive in the 2^{nd} quadrant.

$$\begin{aligned}
x &= \pi-.3747\\
x &= 2.7669\text{radians}
\end{aligned}$$

2.

$$\begin{aligned}
2\ \cos 3x-1 &= 0\\
2\ \cos 3x-1 &= 0 \rightarrow \text{isolate} \cos 3x\\
\frac{2\ \cos 3x}{2} &= \frac{1}{2}\\
\cos 3x &= \frac{1}{2}\\
\cos^{-1}(\cos 3x) &= \cos^{-1}\left(\frac{1}{2}\right)\\
\cos 3x &= \frac{1}{2}
\end{aligned}$$

The interval $[0,2\pi]$ must be tripled since the equation has been solved for $\cos 3x$, The interval is now $[0,6\pi]$. To determine the values for x , each of these values must be divided by 3 .

$$\begin{aligned}
3x &= \frac{\pi}{3},\frac{5\pi}{3},\frac{7\pi}{3},\frac{11\pi}{3},\frac{13\pi}{3},\frac{17\pi}{3}\\
x &= \frac{\pi}{9},\frac{5\pi}{9},\frac{7\pi}{9},\frac{11\pi}{9},\frac{13\pi}{9},\frac{17\pi}{9}
\end{aligned}$$

3.

$$\begin{aligned}
2\sec^2\theta - \tan^4\theta &= -1 \\
2\sec^2\theta - \tan^4\theta &= -1 \rightarrow \sec^2\theta = 1+\tan^2\theta \\
2(1+\tan^2\theta) - \tan^4\theta &= -1 \rightarrow \text{expand} \\
2+2\tan^2\theta - \tan^4\theta &= -1 \rightarrow \text{simplify} \\
2+2\tan^2\theta - \tan^4\theta + 1 &= 0 \rightarrow \text{simplify} \\
-\tan^4\theta + 2\tan^2\theta + 3 &= 0 \rightarrow \div(-1) \\
\tan^4\theta - 2\tan^2\theta - 3 &= 0 \rightarrow \text{factor} \\
(\tan^4\theta + 1)(\tan^2\theta - 3) &= 0 \rightarrow \text{solve}
\end{aligned}$$

Then	Or
$\tan^2\theta + 1 = 0$	$\tan^2\theta - 3 = 0$
$\tan^2\theta = -1$	$\tan^2\theta = 3$
$\sqrt{\tan^2\theta} = \sqrt{-1}$	$\sqrt{\tan^2\theta} = \sqrt{3}$
Does Not Exist	$\tan\theta = \pm\sqrt{3}$
	$\tan^{-1}(\tan\theta) = \tan^{-1}(\pm\sqrt{3})$
	For all real values of θ
	$\theta = \frac{\pi}{3} + \pi k$ and $\theta = -\frac{\pi}{3} + \pi k$ where k is any int eger

4.

$$\begin{aligned}
\sin^2 x - 2 &= \cos 2x \\
\sin^2 x - 2 &= \cos 2x \rightarrow \cos 2x = 1 - 2\sin^2 x \\
\sin^2 x - 2 &= 1 - 2\sin^2 x \rightarrow \text{simplify} \\
\sin^2 x + 2\sin^2 x &= 1 + 2 \rightarrow \text{simplify} \\
3\sin^2 x &= 3 \rightarrow \text{solve} \\
\frac{3\sin^2 x}{3} &= \frac{3}{3} \rightarrow \text{solve} \\
\sin^2 x &= 1 \\
\sqrt{\sin^2 x} &= \pm\sqrt{1} \\
\sin x &= 1 \\
\sin x &= -1
\end{aligned}$$

Over the interval $0^\circ \le x < 360^\circ$

$$\begin{aligned}
\sin^{-1}(\sin x) &= \sin^{-1}(1) \\
x &= 90^\circ \\
\sin^{-1}(\sin x) &= \sin^{-1}(1) \\
x &= 270^\circ
\end{aligned}$$

Solving Trigonometric Equations Using Inverse Notation

Review Exercises

1. To solve $y = \pi - \text{arc sec } 2x$ for x , the restricted range of arcsecant must be considered.

$$\begin{aligned} y &= \pi - \text{arc sec } 2x \\ y &= \pi - \text{arc sec } 2x \rightarrow i\text{solate arc sec } 2x \\ \text{arc sec } 2x &= \pi - y \\ 2x &= \sec(\pi - y) \\ x &= -\frac{1}{2}\sec\ y & \sec(\pi - y) = -\sec\ y \end{aligned}$$

Since the values of arc sec $2x$ are restricted, so are the values of y .

2. To determine the value of $\sin(\cot^{-1}(1))$, the special triangles may be used or technology may be used.

$$\begin{aligned} \cot^{-1}(1) &= \frac{1}{\tan}(1) = 45^\circ \\ \sin(45^\circ) &= \frac{1}{\sqrt{2}} = \left(\frac{\sqrt{2}}{\sqrt{2}}\right) = \frac{\sqrt{2}}{2} \end{aligned}$$

Or

$$\sin\ 45^\circ = 0.7071 \rightarrow \textit{using} \text{ techno log } y$$

3.

$$\begin{aligned} 5\ \cos x - \sqrt{2} &= 3\ \cos x \\ 5\ \cos x - \sqrt{2} &= 3\ \cos x \rightarrow i\text{solare } \cos x \\ 5\ \cos x - 3\ \cos x &= \sqrt{2} \rightarrow \text{simplify} \\ 2\ \cos x &= \sqrt{2} \rightarrow \text{simplify} \\ \frac{2\ \cos x}{2} &= \frac{\sqrt{2}}{2} \rightarrow \text{simplify} \end{aligned}$$

$\cos x = \frac{\sqrt{2}}{2} \rightarrow$ *The graph of the cosine function is one-to-one over the interval* $[0.\pi]$.

If the interval is restricted to $[0.\pi]$, the arccosine of both sides of the equation would give an acceptable result.

$$\begin{aligned} \cos^{-1}(\cos x) &= \cos^{-1}\left(\frac{\sqrt{2}}{2}\right) \\ x &= \frac{\pi}{4} \text{which is within the restricted range of } [0, \pi]. \end{aligned}$$

However, this is the reference angle and the cosine function is also positive in the 4th quadrant. In this quadrant, the result would be $x = 2\pi - \frac{\pi}{4} = \frac{7\pi}{4}$ which is within the interval $[0, 2\pi]$. To include all real solutions which would repeat every 2π units, the solutions for x could be expressed as $x = \frac{\pi}{4} + 2\pi k$ *where k is any int* eger

4.

$$\begin{aligned}
\sec\ \theta - \sqrt{2} &= 0 \\
\sec\ \theta - \sqrt{2} &= 0 \rightarrow i\text{solate } \sec\ \theta \\
\sec\ \theta = \sqrt{2} &\rightarrow \sec\ \theta = \frac{1}{\cos\ \theta} \\
\cos\ \theta &= \frac{1}{\sqrt{2}}
\end{aligned}$$

The graph of the cosine function is one-to-one over the interval $[0, \pi]$. If the interval is restricted to $[0, \pi]$, the arccosine of both sides of the equation would give an acceptable result.

$$\begin{aligned}
\cos^{-1}(\cos\ \theta) &= \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) \\
\theta &= 45^\circ
\end{aligned}$$

However, this is the reference angle and the cosine function is also positive in the 4th quadrant. In this quadrant, the result would be $x = 360^\circ - 45^\circ = 315^\circ$ which is within the interval $0^\circ \leq \theta < 360^\circ$. To include all real solutions which would repeat every 360° , the solutions for x could be expressed as $x = 45^\circ + 360^\circ\ k$ and *where k* is *any integer* and $x = 315^\circ + 360^\circ\ k$ *where k* is *any integer*.

Review Exercises

1. To solve $i = I_m[\sin(wt + \alpha)\cos\ \varphi + \cos(wt + \alpha)\sin\ \varphi]$ for t , the sum formula for sine must be applied.

$$
\begin{aligned}
i &= I_m[\sin(wt+\alpha)\cos\ \varphi+\cos(wt+\alpha)\sin\ \varphi]\\
i &= I_m[\sin(wt+\alpha)\cos\ \varphi+\cos(wt+\alpha)\sin\ \varphi] \to \sin(a+b)=\sin\ a\ \cos\ b+\cos\ a\ \sin\ b\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \to a=wt+\alpha \text{ and } b=\varphi\\
i &= I_m[\sin(wt+\alpha)+(\varphi)] \to \text{simplify}\\
\frac{i}{I_m} &= \frac{I_m[\sin(wt+\alpha)+(\varphi)]}{I_m} \to \text{simplify}\\
\frac{i}{I_m} &= \frac{\cancel{I_m}[\sin(wt+\alpha)+(\varphi)]}{\cancel{I_m}} \to \text{simplify}\\
\frac{i}{I_m} &= \sin(wt+\alpha)+(\varphi) \to \text{simplify}\\
\frac{i}{I_m}-\alpha-\varphi &= \sin\ wt+\alpha-a+\varphi-\varphi \to \text{simplify}\\
\left(\frac{i}{I_m}-\alpha-\varphi\right) &= \sin\ wt \to \div(w)\\
\frac{1}{w}\left(\frac{i}{I_m}-\alpha-\varphi\right) &= \frac{\sin\ \cancel{wt}}{\cancel{w}} \to \text{simplify}\\
\frac{1}{w}\left(\frac{i}{I_m}-\alpha-\varphi\right) &= \sin\ t \to \text{solve}\\
\frac{1}{w}\sin^{-1}\left(\frac{i}{I_m}-\alpha-\varphi\right) &= \sin^{-1}(\sin\ t)\\
\frac{1}{w}\sin^{-1}\left(\frac{i}{I_m}-\alpha-\varphi\right) &= t
\end{aligned}
$$

Review Exercises

1. Solving the following equation will not produce a numerical answer but it will result in an expression that is equal to theta.

$$
\begin{aligned}
I &= I_0 \ \sin\ 2\theta\ \cos\ 2\theta \\
I &= I_0 \ \sin\ 2\theta\ \cos\ 2\theta \rightarrow I_0 \\
\frac{I}{I_0} &= \frac{\cancel{I_0}\ \sin\ 2\theta\ \cos\ 2\theta}{\cancel{I_0}} \rightarrow \text{simplify} \\
\frac{I}{I_0} &= \sin\ 2\theta\ \cos\ 2\theta \rightarrow \times(2) \\
2\left(\frac{I}{I_0}\right) &= 2(\sin\ 2\theta\ \cos\ 2\theta) \rightarrow \text{simplify} \\
\frac{2I}{I_0} &= \sin\ 4\theta \rightarrow \text{solve} \\
\sin^{-1}\left(\frac{2I}{I_0}\right) &= \sin^{-1}(\sin\ 4\theta) \rightarrow \text{simplify} \\
\sin^{-1}\left(\frac{2I}{I_0}\right) &= 4\theta \rightarrow \div(4) \\
\sin^{-1}\left(\frac{2I}{I_0}\right) &= 4\theta \rightarrow \div(4) \\
\left(\frac{1}{4}\right)\sin^{-1}\left(\frac{2I}{I_0}\right) &= \frac{\cancel{4}\ \theta}{\cancel{4}} \\
\left(\frac{1}{4}\right)\sin^{-1}\left(\frac{2I}{I_0}\right) &= \theta
\end{aligned}
$$

2. At first glance, it seems that the diagram does not provide enough information. In order to obtain the answer, various values for theta will have to be substituted into the volume formula to determine when the maximum volume occurs. This question would be a great group activity. The volume of the trough is 10 times the area of the end of the trough. The end of the trough consists of two identical right triangles. The area of each triangle is $\frac{1}{2}(\sin\ \theta)(\cos\ \theta)$. The area of both triangles is $2\left(\frac{1}{2}(\sin\ \theta)(\cos\ \theta)\right) = (\sin\ \theta)(\cos\ \theta)$. The area of the rectangle is $(1)(\cos\ \theta)$. The angles of the rectangle are 90° and the angles of the right triangles must be less than 90° . Therefore, the values of theta that must be considered are $0 \leq \theta \leq \frac{\pi}{2}$.

The formula for the total volume of the trough is:

$$
\begin{aligned}
V &= 10(\sin\ \theta\ \cos\ \theta + \cos\ \theta) \text{ or} \\
V &= 10(\cos\ \theta)(\sin\ \theta + 1)
\end{aligned}
$$

As values for theta are substituted into the formula, the calculated results must be recorded. The maximum volume is 13 ft^3 and occurs when $\theta = \frac{\pi}{6}(30^\circ)$.

CHAPTER 5 Triangles and Vectors - Solution Key

Chapter Outline

5.1 Triangles and Vectors

The Law of Cosines

Review Exercises:

1. a) Using the two given sides and the included angle, the Law of Cosines must be used to calculate the length of side a .

b) Using the lengths of the three given sides, the Law of Cosines must be used to calculate the measure of each of the three angles of $\triangle IRT$.

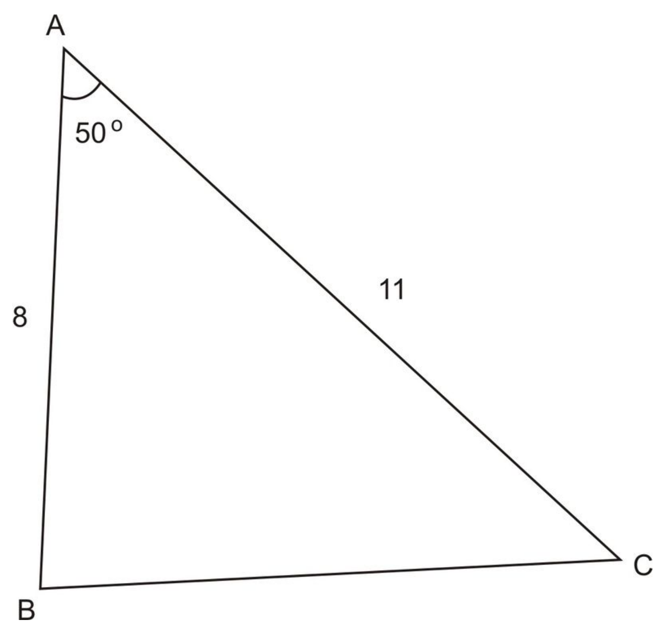
c) Using the two given sides and the included angle of $\triangle PLM$ the Law of Cosines must be used to calculate the length of side $l(PM)$.

d) Using the lengths of the three given sides, the Law of Cosines must be used to determine the measure of the two remaining angles - $\angle R \ and \ \angle D$.

e) Using the two given sides and the included angle, the Law of Cosines must be used to calculate the length of side b .

f) Using the lengths of the three given sides, the Law of Cosines must be used to calculate the measure of each of the three angles of $\triangle CDM$.

2. Given:



$$\angle A = 50^\circ,\ b = 8,\ c = 11$$

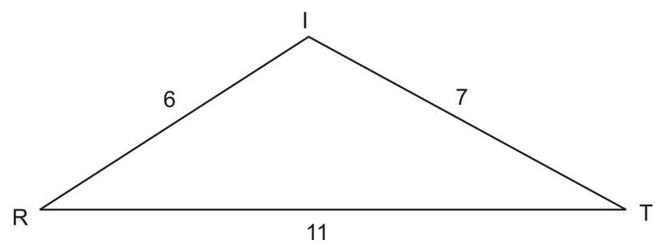
The length of side a can be determined by using the Law of Cosines.

$$a^2 = b^2 + c^2 - 2bc\cos A$$
$$a^2 = b^2 + c^2 - 2bc\cos A \rightarrow \angle A = 50^\circ,\ b = 8,\ c = 11$$
$$a^2 = (8)^2 + (11)^2 - 2(8)(11)\cos 50^\circ \rightarrow \text{simplify}$$

This can be entered into the calculator, as shown, in one step. Press enter when complete.

$$a^2 = 71.8693807 \rightarrow \sqrt{\text{both sides}}$$
$$\sqrt{a^2} = \sqrt{71.8693807} \rightarrow \sqrt{\text{both sides}}$$
$$a \approx 8.48 \text{ units}$$

b) Given:

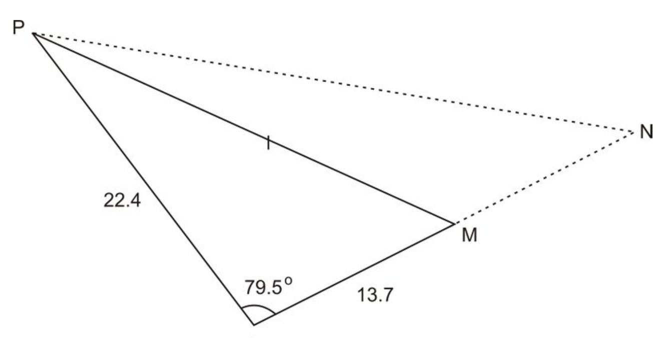


$$i = 11,\ r = 7,\ t = 6$$

The largest angle is across from the longest side. Therefore, determine the measure of $\angle I$ using the Law of Cosines.

$$\cos\angle I = \frac{r^2 + t^2 - i^2}{2rt}$$
$$\cos\angle I = \frac{r^2 + t^2 - i^2}{2rt} \rightarrow i = 11,\ r = 7,\ t = 6$$
$$\cos\angle I = \frac{(7)^2 + (6)^2 + (11)^2}{2(7)(6)} \rightarrow \text{express answer as a fraction}$$
$$\cos\angle I = \frac{-36}{84} \rightarrow \text{divide}$$
$$\cos\angle I = -0.4286 \rightarrow \text{A negative indicates that the angle is greater than } 90^\circ.$$
$$\cos^{-1}(\cos\angle I) = \cos^{-1}(-0.4286)$$
$$\angle I \approx 115.4^\circ$$

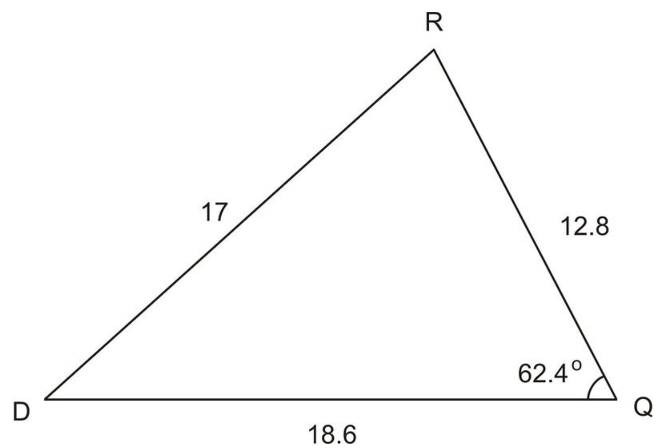
c) Given:



$$\angle L = 79.5^\circ,\ m = 22.4,\ p = 13.7$$

$$\begin{aligned}
l^2 &= m^2 + p^2 - 2mp\ \cos L \\
l^2 &= m^2 + p^2 - 2mp\ \cos L \to \angle L = 79.5^\circ,\ m = 22.4,\ p = 13.7 \\
l^2 &= (22.4)^2 + (13.7)^2 - 2(22.4)(13.7)\ \cos(79.5^\circ) \to \text{simplify} \\
l^2 &= 577.6011 \to \sqrt{\text{both sides}} \\
\sqrt{i^2} &= \sqrt{577.6011} \\
l &\approx 24.03 \text{ units}
\end{aligned}$$

d) Given:

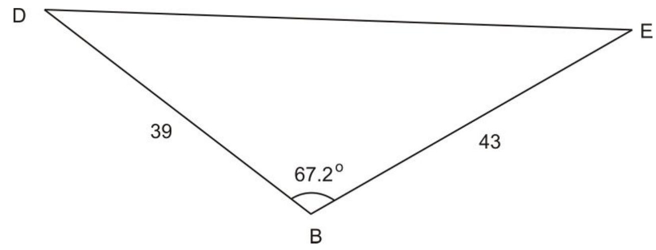


$$d = 12.8,\ q = 17,\ r = 18.6,\ \angle Q = 62.4^\circ$$

The smallest angle is across from the shortest side. Therefore, determine the measure of $\angle D$ using the Law of Cosines.

$$\begin{aligned}
\cos\angle D &= \frac{q^2 + r^2 - d^2}{2qr} \\
\cos\angle D &= \frac{q^2 + r^2 - d^2}{2qr} \to d = 12.8,\ q = 17,\ r = 18.6 \\
\cos\angle D &= \frac{(17)^2 + (18.6)^2 - (12.8)^2}{2(17)(18.6)} \to \text{simplify} \\
\cos\angle D &= \frac{471.12}{632.4} \to \text{divide} \\
\cos\angle D &= 0.7450 \\
\cos^{-1}(\cos\angle D) &= \cos^{-1}(0.7450) \\
\angle D &\approx 41.8^\circ
\end{aligned}$$

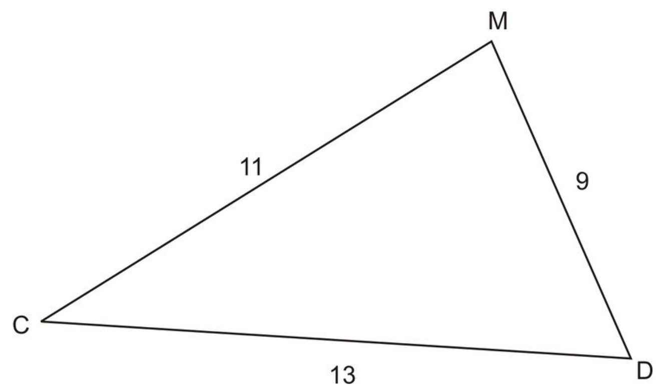
e) Given:



$$d = 43,\ e = 39,\ \angle B = 67.2^\circ$$

$$\begin{aligned}
b^2 &= d^2 + e^2 - 2de\ \cos B \\
b^2 &= d^2 + e^2 - 2de\ \cos B \rightarrow d = 43,\ e = 39,\ \angle B = 67.2^\circ \\
b^2 &= (43)^2 + (39)^2 - 2(43)(39)\ \cos(67.2^\circ) \rightarrow \text{simplify} \\
b^2 &= 2070.2727 \rightarrow \sqrt{\text{both sides}} \\
b &\approx 45.5 \text{ units}
\end{aligned}$$

f) Given:



$$c = 9,\ d = 11,\ m = 13$$

The second largest angle is across from the second longest side. Therefore, determine the measure of $\angle D$ using the Law of Cosines.

$$\begin{aligned}
\cos \angle D &= \frac{c^2 + m^2 - d^2}{2cm} \\
\cos \angle D &= \frac{c^2 + m^2 - d^2}{2cm} \rightarrow c = 9,\ d = 11,\ m = 13 \\
\cos \angle D &= \frac{(9)^2 + (13)^2 - (11)^2}{2(9)(13)} \rightarrow \text{simplify} \\
\cos \angle D &= \frac{129}{234} \rightarrow \text{divide} \\
\cos \angle D &= 0.5513 \\
\cos^{-1}(\cos \angle D) &= \cos^{-1}(0.5513) \\
\angle D &\approx 56.5^\circ
\end{aligned}$$

3. Given $\triangle CIR$ with $c = 63,\ i = 52,\ r = 41.9$. The Law of Cosines may be used to determine the measure of two of the angles and then the third can be determined by subtracting their sum from 180° .

$$\begin{aligned}
\cos \angle C &= \frac{i^2+r^2-c^2}{2ir} \\
\cos \angle C &= \frac{i^2+r^2-c^2}{2ir} \rightarrow c = 63,\ i = 52,\ r = 41.9 \\
\cos \angle C &= \frac{(52)^2+(41.9)^2-(63)^2}{2(52)(41.9)} \rightarrow \text{simplify} \\
\cos \angle C &= \frac{490.61}{4357.6} \rightarrow \text{divide} \\
\cos \angle C &= 0.1123 \\
\cos^{-1}(\cos \angle C) &= \cos^{-1}(0.1123) \\
\angle C &\approx 83.5^\circ
\end{aligned}$$

$$\begin{aligned}
\cos \angle I &= \frac{c^2+r^2-i^2}{2cr} \\
\cos \angle I &= \frac{c^2+r^2-i^2}{2cr} \rightarrow c = 63,\ i = 52,\ r = 41.9 \\
\cos \angle I &= \frac{(63)^2+(41.9)^2-(52)^2}{2(63)(41.9)} \rightarrow \text{simplify} \\
\cos \angle I &= \frac{3020.61}{5279.4} \rightarrow \text{divide} \\
\cos \angle I &= 0.5721 \\
\cos^{-1}(\cos \angle I) &= \cos^{-1}(0.5721) \\
\angle I &\approx 55.1^\circ
\end{aligned}$$

$$\begin{aligned}
\angle R &\approx 180^\circ - (83.5^\circ + 55.1^\circ) \\
\angle R &\approx 41.4^\circ
\end{aligned}$$

4. There are many ways to determine the length of AD . One way is to simply apply the trigonometric ratios.

In $\triangle BCD$:

$$\begin{aligned}
\cos \angle C &= \frac{\text{adj}}{\text{hyp}} \\
\cos(37.4^\circ) &= \frac{x}{14.2} \\
0.7944 &= \frac{x}{14.2} \\
(14.2)0.7944 &= \cancel{(14.2)}\frac{x}{\cancel{14.2}} \\
11.3 \text{ units} &\approx x
\end{aligned}$$

$$\overline{AD} = \overline{AC} - \overline{CD}$$
$$\overline{AD} = 15 - 11.3$$
$$\overline{AD} \approx 3.7 \text{ units}$$

5. In $\triangle HIK \rightarrow HI = 6.7,\ IK = 5.2,\ \angle HIK = 96.3^\circ$. The Law of Cosines may be used to determine the length of HK .

$$\begin{aligned}
i^2 &= h^2 + k^2 - 2hk\ \cos I \\
i^2 &= h^2 + k^2 - 2hk\ \cos I \rightarrow HI(k) = 6.7,\ IK(h) = 5.2,\ \angle HIK(\angle I) = 96.3^\circ \\
i^2 &= (5.2)^2 + (6.7)^2 - 2(5.2)(6.7)\ \cos(96.3^\circ) \rightarrow \text{simplify} \\
i^2 &= 79.5763 \rightarrow \sqrt{\text{both sides}} \\
\sqrt{i^2} &= \sqrt{79.5763} \\
i &= 8.9 \text{ units}
\end{aligned}$$

6. a) In $\triangle ABC \rightarrow a = 20.9,\ b = 17.6,\ c = 15$. The Law of Cosines may be used to confirm the measure of $\angle B$.

$$\begin{aligned}
\cos\angle B &= \frac{a^2 + c^2 - b^2}{2ac} \\
\cos\angle B &= \frac{a^2 + c^2 - b^2}{2ac} \rightarrow a = 20.9,\ b = 17,\ c = 15 \\
\cos\angle B &= \frac{(20.9)^2 + (15)^2 - (17.6)^2}{2(20.9)(15)} \rightarrow \text{simplify} \\
\cos\angle B &= \frac{352.05}{627} \rightarrow \text{divide} \\
\cos\angle B &= 0.5615 \\
\cos^{-1}(\cos\angle B) &= \cos^{-1}(0.5615) \\
\angle B &\approx 55.8^\circ
\end{aligned}$$

$\triangle ABC$ is drawn accurately.

b) In $\triangle DEF \rightarrow d = 16.8,\ e = 24,\ f = 12$. The Law of Cosines may be used to confirm the measure of $\angle D$.

$$\begin{aligned}
\cos\angle D &= \frac{e^2 + f^2 - d^2}{2ef} \\
\cos\angle D &= \frac{e^2 + f^2 - d^2}{2ef} \rightarrow d = 16.8,\ e = 24,\ f = 12 \\
\cos\angle D &= \frac{(24)^2 + (12)^2 - (16.8)^2}{2(4)(12)} \rightarrow \text{simplify} \\
\cos\angle D &= \frac{437.76}{576} \rightarrow \text{divide} \\
\cos\angle D &= 0.76 \\
\cos^{-1}(\cos\angle D) &= \cos^{-1}(0.76) \\
\angle D &\approx 40.5^\circ
\end{aligned}$$

The Law of Cosines may now be applied to determine the correct length of side d .

$$\begin{aligned}
d^2 &= e^2+f^2-2ef\ \cos D\\
d^2 &= e^2+f^2-2ef\ \cos D \rightarrow \angle D=30^\circ,\ e=24,\ f=12\\
d^2 &= (24)^2+(12)^2-2(24)(12)\ \cos(30^\circ) \rightarrow \text{simplify}\\
d^2 &= 221.1694 \rightarrow\ \sqrt{\text{both sides}}\\
\sqrt{d^2} &= \sqrt{221.1694}\\
d &= 14.9 \text{ units}
\end{aligned}$$

$\triangle DEF$ is not accurately drawn. The length of side d is off by approximately $16.8-14.9=1.9$ units .

7. To determine how long the cell phone service will last, the distance must be calculated and then this distance will have to be divided by the speed of the vehicle. The Law of Cosines may be used to calculate the distance.

$$\begin{aligned}
d^2 &= e^2+f^2-2ef\ \cos D\\
d^2 &= e^2+f^2-2ef\ \cos D \rightarrow e=31 \text{ m},\ f=26 \text{ m},\ \angle D=47^\circ\\
d^2 &= (31)^2+(26)^2-2(31)(26)\ \cos(47^\circ) \rightarrow \text{simplify}\\
d^2 &= 537.6186 \rightarrow\ \sqrt{\text{both sides}}\\
\sqrt{d^2} &= \sqrt{537.6186}\\
d &= 23.2 \text{ m}
\end{aligned}$$

To determine the length of time that the cell phone service will last, divide this distance by the speed of 45 mph

$$\frac{23.2\ \cancel{m}}{45\ \cancel{m}/h} \approx 0.52 \text{ hours} \approx 31.2 \text{ minutes}$$

If the answer for the distance is not rounded to 23.2 m as well as the answer for the number of hours, then the cell phone service will last approximately 30.9 minutes.

$$\frac{23.18660483}{45} = 0.5152578851 \approx (0.5152578851)(60) \approx 30.9 \text{ minutes}$$

b) $\frac{23.2\ \cancel{m}}{35\ \cancel{m}/h} \approx 0.66 \text{ hours} \approx 39.6 \text{ minutes}$

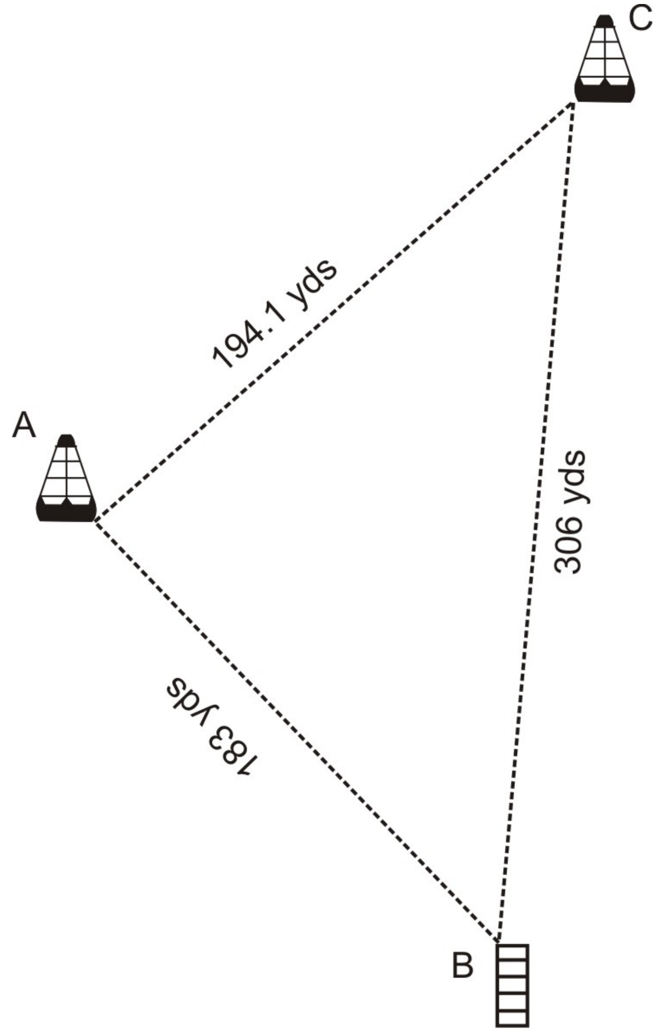
If the speed is reduced to 35 mph , the cell phone service will last for approximately 39.6 minutes which is 8.4 minutes longer.

Or

$$\frac{23.18660483}{35} = 0.6624744237 \approx (0.6624744237)(60) \approx 39.7 \text{ minutes}$$

In this case, the cell phone service will last 8.8 minutes longer.

8. a)



$$
\begin{aligned}
\cos \angle B &= \frac{a^2+c^2-b^2}{2ac} \\
\cos \angle B &= \frac{a^2+c^2-b^2}{2ac} \rightarrow a = 306,\ b = 194.1,\ c = 183 \\
\cos \angle B &= \frac{(306)^2+(183)^2-(194.1)^2}{2(306)(183)} \rightarrow \text{simplify} \\
\cos \angle B &= \frac{89450.19}{111996} \rightarrow \text{divide} \\
\cos \angle B &= 0.7687 \\
\cos^{-1}(\cos \angle B) &= \cos^{-1}(07987) \\
\angle B &\approx 37^\circ
\end{aligned}
$$

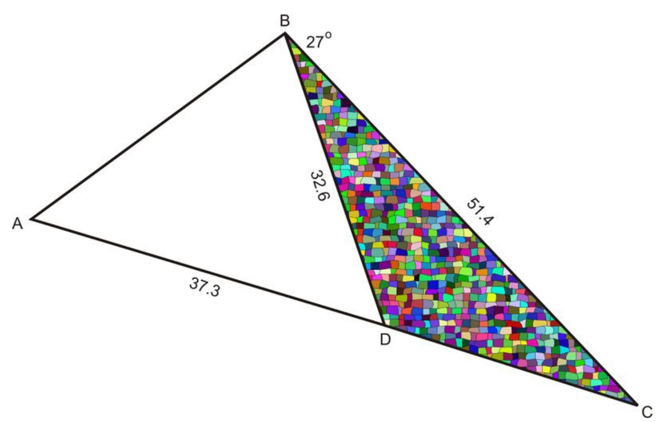
The dock forms an angle of 37° with the two buoys.

b)

$$\begin{aligned}
\cos \angle B &= \frac{a^2+c^2-b^2}{2ac} \\
\cos \angle B &= \frac{a^2+c^2-b^2}{2ac} \rightarrow a=329,\ b=207,\ c=183 \\
\cos \angle B &= \frac{(329)^2+(183)^2-(207)^2}{2(329)(183)} \rightarrow \text{simplify} \\
\cos \angle B &= \frac{98881}{120414} \rightarrow \text{divide} \\
\cos \angle B &= 0.8212 \\
\cos^{-1}(\cos \angle B) &= \cos^{-1}(0.8212) \\
\angle B &\approx 34.8^\circ
\end{aligned}$$

If the distance from the second buoy to both the dock and the first buoy is increased, the dock makes a smaller angle with the two buoys. The angle is 34.8° and is 2.2° smaller.

9.



In $\triangle BCD$, the Law of Cosines may be used to determine the length of DC (b) and then again to calculate the measure of $\angle C$. To determine the length of AB , the Law of Cosines can be used once again with $\triangle ABC$.

In $\triangle BCD$:

$$\begin{aligned}
b^2 &= c^2+d^2-2cd\ \cos B \\
b^2 &= c^2+d^2-2cd\ \cos B \rightarrow c=32.6,\ d=51.4,\ \angle B=27^\circ \\
b^2 &= (32.6)^2+(51.4)^2-2(32.6)(51.4)\ \cos(77^\circ) \rightarrow \text{simplify} \\
b^2 &= 718.7077 \rightarrow \sqrt{\text{both sides}} \\
\sqrt{b^2} &= \sqrt{718.7077} \\
b &= 26.8 \text{ feet}
\end{aligned}$$

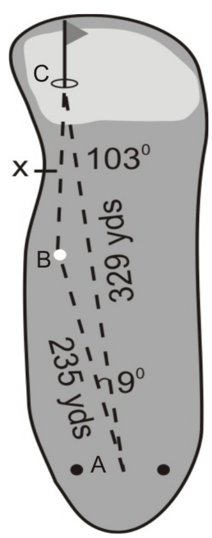
$$\cos\angle C = \frac{b^2+d^2-c^2}{2bd}$$
$$\cos\angle C = \frac{b^2+d^2-c^2}{2bd} \rightarrow b = 26.8,\ c = 32.6,\ d = 51.4$$
$$\cos\angle C = \frac{(26.8)^2+(51.4)^2-(32.6)^2}{2(26.8)(51.4)} \rightarrow \text{simplify}$$
$$\cos\angle C = \frac{2297.44}{2755.04} \rightarrow \text{divide}$$
$$\cos\angle C = 0.8339$$
$$\cos^{-1}(\cos\angle C) = \cos^{-1}(0.8339)$$
$$\angle C \approx 33.5^\circ$$

In $\triangle ABC,\ a = 51.4,\ b = 37.3+26.8 = 64.1,\ \angle C = 33.5^\circ$.

$$c^2 = a^2+b^2-2ab\cos C$$
$$c^2 = a^2+b^2-2ab\cos C \rightarrow a = 51.4,\ b = 64.1,\ \angle C = 33.5^\circ$$
$$c^2 = (51.4)^2+(64.1)^2-2(51.4)(64.1)\ \cos(33.5^\circ) \rightarrow \text{simplify}$$
$$c^2 = 1255.8961 \rightarrow \sqrt{\text{both sides}}$$
$$\sqrt{c^2} = \sqrt{1255.8961}$$
$$c \approx 35.4 \text{ feet}$$

The length of AB is not 34.3 feet . It is 35.4 feet.

10.



a) To determine the distance that the ball is from the hole, use the Law of Cosines to find the length of side a .

In $\triangle ABC \rightarrow b = 329,\ c = 235,\ \angle A = 9^\circ$

$$
\begin{aligned}
a^2 &= b^2+c^2-2bc\cos A \\
a^2 &= b^2+c^2-2bc\cos A \rightarrow b=329,\ c=235,\ \angle A=9^\circ \\
a^2 &= (329)^2+(235)^2-2(329)(235)\ \cos(9^\circ) \rightarrow \text{simplify} \\
a^2 &= 10739.7519 \rightarrow \sqrt{\text{both sides}} \\
\sqrt{a^2} &= \sqrt{10739.7519} \\
a &\approx 103.6 \text{ yds} \\
a &\approx 103.6 \text{ yards}
\end{aligned}
$$

b) No solution.

11. There answers to this question are numerous. Below is one example of a possible solution.

Three towns, A, B, and C respectively, are separated by distances that form a triangle. Town A is 127 miles from Town B and Town B is 210 miles from Town C . If the angle formed at Town B is 17° , calculate the number miles you would have to travel to complete a round trip that Begins at Town A .

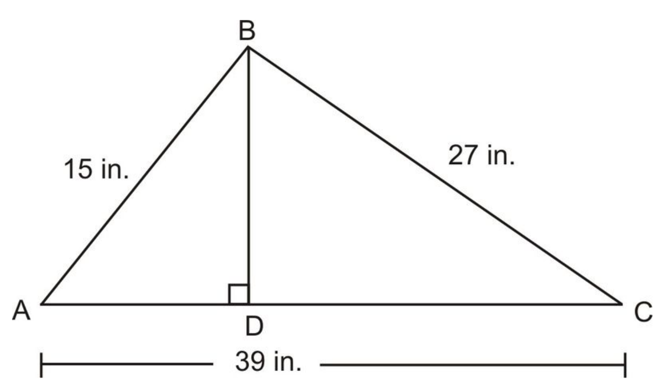
To answer this problem, the distance between Town A and Town C must be determined. Then the three distances must be added to determine the length of a round trip.

$$
\begin{aligned}
b^2 &= a^2+c^2-2ac\ \cos B \\
b^2 &= a^2+c^2-2ac\ \cos B \rightarrow a=127,\ c=210,\ \angle B=17^\circ \\
b^2 &= (127)^2+(210)^2-2(127)(210)\ \cos(17^\circ) \rightarrow \text{simplify} \\
b^2 &= 9219.7043 \rightarrow \sqrt{\text{both sides}} \\
\sqrt{b^2} &= \sqrt{9219.7043} \\
b &\approx 96.0 \text{ miles}
\end{aligned}
$$

The distance you would travel to complete a round trip that begins in Town A is $127+210+96=433$ miles .

12. There answers to this question are numerous. Below is one example of a possible solution.

Given $\triangle ABC$, calculate the area of the triangle to the nearest tenth.



In $\triangle ABC$, the Law of Cosines may be used to calculate the measure of $\angle A$ or $\angle C$.

$$\begin{aligned}
\cos \angle A &= \frac{b^2+c^2-a^2}{2bc} \\
\cos \angle A &= \frac{b^2+c^2-a^2}{2bc} \rightarrow a=27,\ b=39,\ c=15 \\
\cos \angle A &= \frac{(39)^2+(15)^2-(27)^2}{2(39)(15)} \rightarrow \text{simplify} \\
\cos \angle A &= \frac{1017}{1170} \rightarrow \text{divide} \\
\cos \angle A &= 0.8692 \\
\cos^{-1}(\cos \angle A) &= \cos^{-1}(0.8692) \\
\angle A &\approx 29.6^\circ
\end{aligned}$$

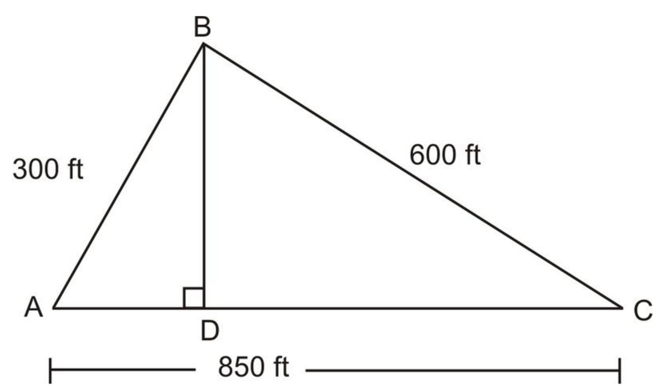
Use the sine ratio to calculate the height of the altitude.

$$\begin{aligned}
\sin \angle A &= \frac{\text{opp}}{\text{hyp}} \\
\sin 29.6^\circ &= \frac{x}{15} \\
0.4939 &= \frac{x}{15} \\
(15)(0.4939) &= (\cancel{15})\left(\frac{x}{\cancel{15}}\right) \\
7.4 \text{ inches} &\approx x
\end{aligned}$$

The area of the triangle is $\frac{1}{2} b \cdot h$

$$\begin{aligned}
\text{Area} &= \frac{1}{2} b \cdot h \rightarrow b=39,\ h=7.4 \\
\text{Area} &= \frac{1}{2}(39)\cdot(7.4) \rightarrow \text{simplify} \\
\text{Area} &= 144.3 \text{ in}^2
\end{aligned}$$

13. This question is similar to the one above. The additional step is to divide the area by 42000 ft^2 to determine the number of acres of land.



In $\triangle ABC$, the Law of Cosines may be used to calculate the measure of $\angle A$ or $\angle C$.

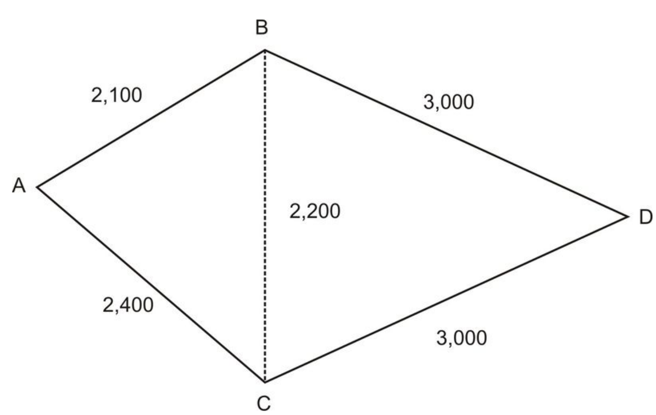
$$\begin{aligned}
\cos\angle A &= \frac{b^2+c^2-a^2}{2bc} \\
\cos\angle A &= \frac{b^2+c^2-a^2}{2bc} \rightarrow a=600,\ b=850,\ c=300 \\
\cos\angle A &= \frac{(850)^2+(300)^2-(600)^2}{2(850)(300)} \rightarrow \text{simplify} \\
\cos\angle A &= \frac{452500}{510000} \rightarrow \text{divide} \\
\cos\angle A &= 0.8873 \\
\cos^{-1}(\cos\angle A) &= \cos^{-1}(0.8873) \\
\angle A &\approx 27.5^\circ
\end{aligned}$$

Use the sine ratio to calculate the height of the altitude.

$$\begin{aligned}
\sin\angle A &= \frac{\text{opp}}{\text{hyp}} \\
\sin 27.5^\circ &= \frac{x}{300} \\
0.4617 &= \frac{x}{300} \\
(300)(0.4617) &= (\cancel{300})\left(\frac{x}{\cancel{300}}\right) \\
138.5 \text{ feet} &\approx x
\end{aligned}$$

$$\begin{aligned}
&\text{The area of the triangle is } \frac{1}{2}b\cdot h \\
&\text{Area} = \frac{1}{2}b\cdot h \rightarrow b=850,\ h=138.5 \\
&\text{Area} = \frac{1}{2}(850)\cdot(138.5) \rightarrow \text{simplify} \\
&\text{Area} = 58862.5 \text{ ft}^2 \\
&\text{of acres} = \frac{58862.5}{42000} \approx 1.4 \text{ acres}
\end{aligned}$$

14. To determine the area of this quadrilateral, the area of triangles $\triangle ABC$ *and* $\triangle BCD$, must be determined by using the Law of Cosines and the formula Area $= \frac{1}{2}b\cdot h$. The area of each triangle must then be added to obtain the total area of the farm plot.



In $\triangle ABC \rightarrow a=2200,\ b=2400,\ c=2100$. The Law of cosines may be used to determine the measure of one of the angles of the triangle.

$$\begin{aligned}
\cos \angle B &= \frac{a^2+c^2-b^2}{2ac} \\
\cos \angle B &= \frac{a^2+c^2-b^2}{2ac} \rightarrow a = 2200,\ b = 2400,\ c = 2100 \\
\cos \angle B &= \frac{(2200)^2+(2100)^2-(2400)^2}{2(2200)(2100)} \rightarrow \text{simplify} \\
\cos \angle B &= \frac{3490000}{9240000} \rightarrow \text{divide} \\
\cos \angle B &= 0.3777 \\
\cos^{-1}(\cos \angle B) &= \cos^{-1}(0.3777) \\
\angle B &\approx 67.8^\circ
\end{aligned}$$

The length of the altitude drawn from A to BC can be calculated by using the sine ratio.

$$\begin{aligned}
\sin \angle B &= \frac{\text{opp}}{\text{hyp}} \\
\sin 67.8^\circ &= \frac{x}{2100} \\
0.9259 &= \frac{x}{2100} \\
(2100)(0.9259) &= (\cancel{2100})\left(\frac{x}{\cancel{2100}}\right) \\
1944.4 \text{ feet} &\approx x
\end{aligned}$$

$$\begin{aligned}
\text{Area} &= \frac{1}{2}b \cdot h \rightarrow b = 2200,\ h = 1944.4 \\
\text{Area} &= \frac{1}{2}(2200)(1944.4) \rightarrow \text{simplify} \\
\text{Area} &= 2,138,840 \text{ ft}^2
\end{aligned}$$

In $\triangle BCD \rightarrow b = 3000,\ c = 3000, d = 2200$. The Law of cosines may be used to determine the measure of one of the angles of the triangle.

$$\begin{aligned}
\cos \angle C &= \frac{b^2+d^2-c^2}{2bd} \\
\cos \angle C &= \frac{b^2+d^2-c^2}{2bd} \rightarrow b = 3000,\ c = 3000,\ d = 2200 \\
\cos \angle C &= \frac{(3000)^2+(2200)^2-(3000)^2}{2(3000)(2200)} \rightarrow \text{simplify} \\
\cos \angle C &= \frac{4840000}{13200000} \rightarrow \text{divide} \\
\cos \angle C &= 0.3667 \\
\cos^{-1}(\cos \angle C) &= \cos^{-1}(0.3667) \\
\angle C &\approx 68.5^\circ
\end{aligned}$$

The length of the altitude drawn from D to BC can be calculated by using the sine ratio.

$$\begin{aligned}
\sin \angle C &= \frac{\text{opp}}{\text{hyp}} \\
\sin 68.5^\circ &= \frac{x}{3000} \\
0.9304 &= \frac{x}{300} \\
(3000)(0.9304) &= (\cancel{3000})\left(\frac{x}{\cancel{3000}}\right) \\
2791.3 \text{ feet} &\approx x
\end{aligned}$$

$$\begin{aligned}
\text{Area} &= \frac{1}{2} b \cdot h \rightarrow b = 2200,\ h = 2791.3 \\
\text{Area} &= \frac{1}{2}(2200) \cdot (2791.3) \\
\text{Area} &\approx 3,070,430 \text{ ft}^2
\end{aligned}$$

The total area of the quadrilateral farm plot is approximately: $2,138,840 \text{ ft}^2 + 3,070,430 \text{ ft}^2 = 5,209,270 \text{ ft}^2$

15. To determine the length of the cable at each of the reaches, the Law of Cosines can be used.

a)

$$\begin{aligned}
b^2 &= a^2 + c^2 - 2ac\ \cos B \\
b^2 &= a^2 + c^2 - 2ac\ \cos B \rightarrow a = 20,\ c = 4,\ \angle B = 17^\circ \\
b^2 &= (20)^2 + (4)^2 - 2(20)(4)\ \cos(17^\circ) \rightarrow \text{simplify} \\
b^2 &= 262.9912 \rightarrow \sqrt{\text{both sides}} \\
b &\approx 16.2 \text{ m}
\end{aligned}$$

The cable is approximately 16.2 m long at the crane's lowest reach.

b)

$$\begin{aligned}
b^2 &= a^2 + c^2 - 2ac\ \cos B \\
b^2 &= a^2 + c^2 - 2ac\ \cos B \rightarrow a = 20,\ c = 4,\ \angle B = 82^\circ \\
b^2 &= (20)^2 + (4)^2 - 2(20)(4)\ \cos(82^\circ) \rightarrow \text{simplify} \\
b^2 &= 393.7323 \rightarrow \sqrt{\text{both sides}} \\
b &\approx 19.8 \text{ m}
\end{aligned}$$

The cable is approximately 19.8 m long at the crane's highest reach.

16. To solve this problem the Law of Cosines will have to used to determine the length of AB and then used gain to calculate the measure of $\angle AEH$.

$$\begin{aligned}
e^2 &= a^2 + b^2 - 2ab \cos E \\
e^2 &= a^2 + b^2 - 2ab \cos E \rightarrow a = 4,\ b = 21,\ \angle E = 120^\circ \\
e^2 &= (4)^2 + (21)^2 - 2(4)(21)\ \cos(120^\circ) \rightarrow \text{simplify} \\
e^2 &= 541 \rightarrow \sqrt{\text{both sides}} \\
e &\approx 23.3 \text{ cm}
\end{aligned}$$

The length of AB is reduced by 5 cm . when the fluid is pumped out of the cylinder. As a result, The length of $23.3 - 5.0 = 18.3$ cm must be used to calculate the measure of $\angle AEH$.

$$\begin{aligned}
\cos\angle E &= \frac{a^2+b^2-e^2}{2ab}\\
\cos\angle E &= \frac{a^2+b^2-e^2}{2ab} \rightarrow a=4,\ b=21,\ e=18.3\\
\cos\angle E &= \frac{(4)^2+(21)^2-(18.3)^2}{2(4)(21)} \rightarrow \text{simplify}\\
\cos\angle E &= \frac{122.11}{168} \rightarrow \text{divide}\\
\cos\angle E &= 0.7268\\
\cos^{-1}(\cos\angle E) &= \cos^{-1}(0.7268)\\
\angle E &\approx 43.4^\circ
\end{aligned}$$

Area of a Triangle

Review Exercises:

1. a) In $\triangle COM$, Pythagorean Theorem can be used to determine the height of the altitude OF . Then, the length of the base can be calculated by adding the given lengths of CF and FM . With these two measurements, the formula $A=\frac{1}{2}b\cdot h$ may be used to obtain the area of the triangle.

b) The area of $\triangle CEH$ can be calculated by applying Heron's Formula since the length of the each side of the triangle is given.

c) In $\triangle APH$, the length of two sides is given as well as the measure of the included angle. The $K=\frac{1}{2}bc\sin A$ formula may be used to determine the area of the triangle.

d) In $\triangle XLR$, the tangent ratio can be used to determine the height of the altitude LX .

Then, the length of the base can be calculated by adding the given lengths of RX and XE . With these two measurements, the formula $A=\frac{1}{2}b\cdot h$ may be used to obtain the area of the triangle.

2. a) In $\triangle COF$

$$\begin{aligned}
(h)^2 &= (s_1)^2+(s_2)^2\\
(5)^2 &= (3)^2+(s_2)^2\\
25-9 &= s^2\\
\sqrt{16} &= \sqrt{s^2}\\
4 &= s \text{ (OF)}
\end{aligned}$$

$$\begin{aligned}
\text{Base}(\overline{CM}) &= \overline{CF}+\overline{FM})\\
(\overline{CM}) &= 3+8=11
\end{aligned}$$

$$\begin{aligned}
A &= \frac{1}{2}b\cdot h\\
A &= \frac{1}{2}(11)\cdot(4)\\
A &= 22 \text{ units}^2
\end{aligned}$$

b)

$$\begin{aligned}
K &= \sqrt{s(s-a)(s-b)(s-c)} \\
K &= \sqrt{s(s-a)(s-b)(s-c)} \rightarrow s = \frac{1}{2}(4.1+7.4+9.6) = 10.55 \\
&\qquad \rightarrow c(a) = 9.6,\ e(b) = 4.1,\ h(c) = 7.4 \\
K &= \sqrt{10.55(10.55-9.6)(10.55-4.1)(10.55-7.4)} \rightarrow \text{simplify} \\
K &= \sqrt{203.6321438} \\
K &= 14.27 \text{ units}^2
\end{aligned}$$

c)

$$\begin{aligned}
K &= \frac{1}{2}bc\ \sin A \\
K &= \frac{1}{2}bc\ \sin A \rightarrow b(a) = (86.3),\ c(h) = 59.8,\ \angle P(A) = 103^\circ \\
K &= \frac{1}{2}(86.3)(59.8)\sin(103^\circ) \rightarrow \text{simplify} \\
K &= 2514.24 \text{ units}^2
\end{aligned}$$

d)

$$\begin{aligned}
\tan\theta &= \frac{\text{opp}}{\text{adj}} \\
\tan 41^\circ &= \frac{x}{11.1} \\
0.8693 &= \frac{x}{11.1} \\
(11.1)(0.8693) &= (\cancel{11.1})\left(\frac{x}{\cancel{11.1}}\right) \\
9.6 &\approx x
\end{aligned}$$

$$\begin{aligned}
\text{Base}(\overline{ER}) &= \overline{RX} + \overline{XE} \\
(\overline{ER}) &= 11.1 + 18.9 = 30
\end{aligned}$$

$$\begin{aligned}
A &= \frac{1}{2}b \cdot h \\
A &= \frac{1}{2}(30)\cdot(9.6) \\
A &= 144 \text{ units}^2
\end{aligned}$$

3. a) In $\triangle ABC$, the area and the length of the base are given. To determine the length of the $A = \frac{1}{2}b \cdot h$ altitude, the formula must be used.

b) In $\triangle ABC$. The area and the lengths of two sides of the triangle are given. To determine the measure of the included angle, the formula $K = \frac{1}{2}b \cdot c \sin A$ must be used.

c) In $\triangle ABD$, the formula $A = \frac{1}{2}b \cdot h$, can be used to determine the length of the altitude. With this measurement calculated, the tangent function can be used to determine the length of CD . The formula $A = \frac{1}{2}b \cdot h$ can now be used to calculate the area of $\triangle ABC$.

4. a)

$$\begin{aligned}
A &= \frac{1}{2}b \cdot h \\
A &= \frac{1}{2}b \cdot h \rightarrow A = 1618.98,\ b = 36.3 \\
1618.98 &= \frac{1}{2}(36.3) \cdot h \rightarrow \text{simplify} \\
1618.98 &= 18.15h \rightarrow \text{solve} \\
\frac{1618.98}{18.15} &= \frac{\cancel{18.15}}{\cancel{18.15}}h \rightarrow \text{solve} \\
89.2 \text{ units} &= h
\end{aligned}$$

b)

$$\begin{aligned}
K &= \frac{1}{2}bc\sin A \\
K &= \frac{1}{2}bc\sin A \ \rightarrow A = 387.6,\ b = 25.6,\ c = 32.9 \\
387.6 &= \frac{1}{2}(25.6)(32.9)\sin A \rightarrow \text{simplify} \\
387.6 &= 421.12\sin A \rightarrow \text{solve} \\
\frac{387.6}{421.12} &= \frac{421.12}{421.12}\sin A \rightarrow \text{solve} \\
0.9204 &= \sin A \\
\sin^{-1}(0.9204) &= \sin^{-1}(\sin A) \\
67^\circ &\approx \angle A
\end{aligned}$$

c)

$$\begin{aligned}
A &= \frac{1}{2}b \cdot h \\
A &= \frac{1}{2}b \cdot h \rightarrow A = 16.96,\ b = 3.2 \\
16.96 &= \frac{1}{2}(3.2) \cdot h \rightarrow \text{simplify} \\
16.96 &= 1.6h \rightarrow \text{solve} \\
\frac{16.96}{1.6} &= \frac{\cancel{1.6}}{\cancel{1.6}}h \\
10.6 \text{ units} &= h
\end{aligned}$$

In $\triangle BCD$:

$$\tan\angle B = \frac{\text{opp}}{\text{adj}} \qquad 1.1750 = \frac{x}{10.6}$$

$$\tan 49.6^\circ = \frac{x}{10.6} \qquad (10.6)(1.1750) = (\cancel{10.6})\left(\frac{x}{\cancel{10.6}}\right) \rightarrow 12.5 \text{ units} \approx x$$

The total length of the base (AC) *is* $3.2 + 12.5 = 15.7$ units . The area of $\triangle ABC$ is

$$
\begin{aligned}
A &= \frac{1}{2} b \cdot h \\
A &= \frac{1}{2} b \cdot h \rightarrow b = 15.7,\ h = 10.6 \\
A &= \frac{1}{2}(15.7) \cdot (10.6) \rightarrow \text{simplify} \\
A &= 83.21 \text{ units}^2
\end{aligned}
$$

5. a) To determine the total area of the exterior of the Pyramid Hotel, Heron's Formula should be used. The four sides are isosceles triangles so the area of one side can be multiplied by 4 to obtain the total area.

$$
\begin{aligned}
s &= \frac{1}{2}(a+b+c) \\
s &= \frac{1}{2}(a+b+c) \rightarrow a = 375,\ b = 375,\ c = 590 \\
s &= \frac{1}{2}(375+375+590) \rightarrow \text{simplify} \\
s &= 670 \text{ feet} \rightarrow \text{one side}
\end{aligned}
$$

Area of one side:

$$
\begin{aligned}
K &= \sqrt{s(s-a)(s-b)(s-c)} \\
K &= \sqrt{s(s-a)(s-b)(s-c)} \rightarrow s = 670,\ a = 375,\ b = 375,\ c = 590 \\
K &= \sqrt{670(670-375)(670-375)(670-590)} \rightarrow \text{simplify} \\
K &= \sqrt{4664540000} \rightarrow \sqrt{\ } \\
K &= 68,297.4 \text{ ft}^2 \rightarrow \text{one side}
\end{aligned}
$$

Total Area: $(4)68,297.4 \text{ ft}^2 = 273,189.6 \text{ ft}^2$

b) The number of gallons of paint that are needed to paint the hotel is:

$$
\frac{273,189.6 \text{ ft}^2}{25 \text{ ft}^2} = 10927.584 \approx 109,28 \text{ gallons}
$$

5.a) The three sides of the triangular section a have been given in the problem. Therefore, Heron's Formula may be used to calculate the area of the section.

$$
\begin{aligned}
s &= \frac{1}{2}(a+b+c) \\
s &= \frac{1}{2}(a+b+c) \rightarrow a = 8.2,\ b = 14.6,\ c = 16.3 \\
s &= \frac{1}{2}(8.2+14.6+16.3) \rightarrow \text{simplify} \\
s &= 19.55 \text{ feet}
\end{aligned}
$$

$$\begin{aligned}
K &= \sqrt{s(s-a)(s-b)(s-c)}\\
K &= \sqrt{s(s-a)(s-b)(s-c)} \rightarrow s = 19.55,\ a = 8.2,\ b = 14.6,\ c = 16.3\\
K &= \sqrt{19.55(19.55-8.2)(19.55-14.6)(19.55-16.3)} \rightarrow \text{simplify}\\
K &= \sqrt{3569.695594} \rightarrow \sqrt{\ }\\
K &= 59.7 \text{ ft}^2
\end{aligned}$$

The number of bundles of shingles that must be purchased is:

$$59.7 \div 33\frac{1}{3} = 1.791 \approx 2$$

b) The shingles will cost $(2)(\$15.45) = \$30.90$

c) The shingles that will go to waste are:

$$\begin{aligned}
2 - 1.791 &= .209\\
(.209)\left(33\frac{1}{3}\right) &\approx 6.97 \text{ ft}^2
\end{aligned}$$

7. a) To determine the area of the section of crops that need to be replanted, the formula $K = \frac{1}{2}bc\sin A$ may be used because the lengths of two sides and the included angle are known..

$$\begin{aligned}
K &= \frac{1}{2}bc\sin A\\
K &= \frac{1}{2}bc\sin A \ \rightarrow b = 186,\ c = 205,\ \angle A = 148^\circ\\
K &= \frac{1}{2}(186)(205)\sin 148^\circ \rightarrow \text{simplify}\\
K &\approx 10,102.9 \text{ yd}^2
\end{aligned}$$

b)

$$\begin{aligned}
K &= \frac{1}{2}bc\sin A\\
K &= \frac{1}{2}bc\sin A \ \rightarrow b = 186,\ c = 288,\ \angle A = 148^\circ\\
K &= \frac{1}{2}(186)(288)\sin 148^\circ \rightarrow \text{simplify}\\
K &\approx 14,193.4 \text{ yd}^2
\end{aligned}$$

The increase in the area that must be replanted is: $10,102.9 \text{ yd}^2 14,\ 193.4 \text{ yd}^2 - 10,\ 102.9 \text{ yd}^2 = 4090.5 \text{ yd}^2$

8. The length of one side of each triangle can be determined by using the formula $K = \frac{1}{2}bc\sin A$. The third side of each triangle can be found by using the Law of Cosines.

The perimeter of the quadrilateral can then be determined by adding the lengths of the sides.

$\triangle DEG$:

$$
\begin{aligned}
K &= \frac{1}{2}dg\sin E \\
K &= \frac{1}{2}dg\sin E \ \rightarrow K = 56.5,\ d = 13.6,\ \angle E = 39^\circ \\
56.5 &= \frac{1}{2}(13.6)g\sin(39^\circ) \rightarrow \text{simplify} \\
56.5 &= \frac{1}{2}(13.6)g(0.6293) \rightarrow \text{simplify} \\
56.5 &= 4.27924g \rightarrow \text{solve} \\
\frac{56.5}{4.27924} &= \frac{\cancel{4.27924}}{\cancel{4.27924}}g \\
13.2 \text{ units} &\approx g
\end{aligned}
$$

$$
\begin{aligned}
e^2 &= d^2 + g^2 - 2dg\cos E \\
e^2 &= d^2 + g^2 - 2dg\cos E \rightarrow d = 13.6,\ g = 13.2,\ \angle E = 39^\circ \\
e^2 &= (13.6)^2 + (13.2)^2 - 2(13.6)(13.2)\cos(39^\circ) \rightarrow \text{simplify} \\
e^2 &= 80.1735 \rightarrow \sqrt{\text{both sides}} \\
\sqrt{e^2} &= \sqrt{80.1735} \\
e &\approx 9.0 \text{ units}
\end{aligned}
$$

$\triangle EFT$

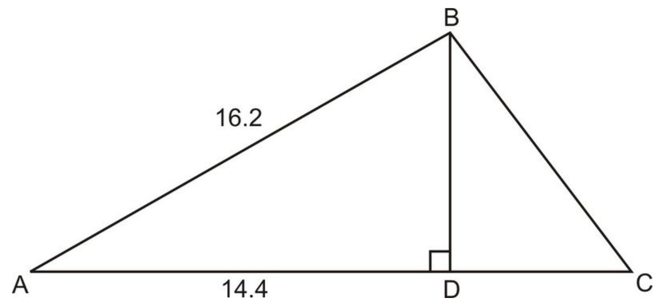
$$
\begin{aligned}
K &= \frac{1}{2}ef\sin G \\
K &= \frac{1}{2}ef\sin G \ \rightarrow K = 84.7,\ f = 13.6,\ \angle G = 60^\circ \\
84.7 &= \frac{1}{2}e(13.6)\sin(60^\circ) \rightarrow \text{simplify} \\
84.7 &= \frac{1}{2}e(13.6)(0.8660) \rightarrow \text{simplify} \\
84.7 &= 5.8888e \rightarrow \text{solve} \\
\frac{84.7}{5.8888} &= \frac{\cancel{5.8888}}{\cancel{5.8888}}e \\
14.4 \text{ units} &\approx e
\end{aligned}
$$

$$
\begin{aligned}
g^2 &= e^2 + f^2 - 2ef\cos G \\
g^2 &= e^2 + f^2 - 2ef\cos G \rightarrow e = 14.4,\ f = 13.6,\ \angle G = 60^\circ \\
g^2 &= (14.4)^2 + (13.6)^2 - 2(14.4)(13.6)\cos(60^\circ) \rightarrow \text{simplify} \\
g^2 &= 196.48 \rightarrow \sqrt{\text{both sides}} \\
\sqrt{g^2} &= \sqrt{196.48} \\
g &\approx 14.0 \text{ units}
\end{aligned}
$$

The perimeter of the quadrilateral is approximately $13.2+9.0+14.4+14.0=50.6$ units .

9. In the following triangle, Pythagorean Theorem can be used to determine the length of the altitude BD . Then the formula $A=\frac{1}{2}b\cdot h$ can be used to determine the length of the base AC .

The difference between the base and AD is the length of DC .



$$\begin{aligned}(h)^2 &= (s_1)^2+(s_2)^2\\(h)^2 &= (s_1)^2+(s_2)^2 \rightarrow h=16.2,\ s_1=14.4\\(16.2)^2 &= (14.4)^2+(s_2)^2 \rightarrow \text{simplify}\\(16.2)^2-(14.4)^2 &= (s_2)^2 \rightarrow \text{simplify}\\55.08 &= (s_2)^2 \rightarrow \sqrt{\text{both sides}}\\\sqrt{55.08} &= \sqrt{(s_2)^2}\\7.4 &\approx s\end{aligned}$$

The altitude of the triangle is approximately 7.4 units.

$$\begin{aligned}A &= \frac{1}{2}b\cdot h\\A &= \frac{1}{2}b\cdot h \rightarrow A=232.96,\ h=7.4\\232.96 &= \frac{1}{2}b\cdot(7.4) \rightarrow \text{simplify}\\232.96 &= 3.7b \rightarrow \text{solve}\\\frac{232.96}{3.7} &= \frac{\cancel{3.7}}{\cancel{3.7}}b \rightarrow \text{solve}\\63.0 \text{ units} &\approx b\end{aligned}$$

$$\begin{aligned}DC &= AC-AD\\DC &= 63.0-14.4=48.6 \text{ units}\end{aligned}$$

10. To show that in any triangle $DEF, d^2+e^2+f^2=2(ef\cos D+df\cos E+de\cos F)$ the Law of Cosines for finding the length of each side, d,e and f will have to be used and the sum of these will have to be simplified.

$$\begin{aligned}
d^2 &= e^2 + f^2 - 2ef\ \cos D \\
e^2 &= d^2 + f^2 - 2df\ \cos E \\
f^2 &= d^2 + e^2 - 2de\ \cos F \\
d^2 + e^2 + f^2 &= (e^2 + f^2 - 2ef \cos D) + (d^2 + f^2 - 2df \cos E) + (d^2 + e^2 - 2de \cos F) \rightarrow \text{simplify} \\
d^2 + e^2 + f^2 &= 2d^2 + 2e^2 + 2f^2 - 2ef \cos D - 2df \cos E - 2de \cos F \rightarrow \text{simplify} \\
d^2 + e^2 + f^2 &= 2d^2 + 2e^2 + 2f^2 - 2ef \cos D - 2df \cos E - 2de \cos F \rightarrow \text{common factor} \\
d^2 + e^2 + f^2 &= 2(d^2 + e^2 + f^2) - 2(ef \cos D + df \cos E + de \cos F) \rightarrow \text{simplify} \\
d^2 + e^2 + f^2 - 2(d^2 + e^2 + f^2) &= -2(ef \cos D + df \cos E + de \cos F) \rightarrow \text{simplify} \\
-(d^2 + e^2 + f^2) &= -2(ef \cos D + df \cos E + de \cos F) \rightarrow \div(-1) \\
d^2 + e^2 + f^2 &= 2(ef \cos D + df \cos E + de \cos F)
\end{aligned}$$

The Law of Sines

Review Exercises:

1. a) This situation represents ASA.

b) This situation represents AAS.

c) This situation represents neither ASA nor AAS. The measure of the 3 angles is given.

d) This situation represents ASA.

e) This situation represents AAS.

f) This situation represents AAS.

2. In all of the above cases, the length of the side across from an angle can be determined. As well, the measure of an angle can be determined.

3. a) To determine the length of side a, the measure of $\angle B$ must be calculated first. This can be done by adding the two given angles and subtracting their sum from 180° . Then the Law of Sines can be used to determine the length of side a .

$$\begin{aligned}
\angle B &= 180^\circ - (11.7^\circ + 23.8^\circ) \\
\angle B &= 180^\circ - (35.5^\circ) \\
\angle B &= 144.5^\circ
\end{aligned}$$

Law of Sines:

$$\begin{aligned}
\frac{a}{\sin A} &= \frac{b}{\sin B} \\
\frac{a}{\sin A} &= \frac{b}{\sin B} \rightarrow b = 16,\ \angle A = 11.7^\circ,\ \angle B = 144.5^\circ \\
\frac{a}{\sin(11.7^\circ)} &= \frac{16}{\sin(144.5^\circ)} \rightarrow \text{simplify} \\
a(\sin(144.5^\circ)) &= 16(\sin(11.7^\circ)) \rightarrow \text{simplify} \\
a(0.5807) &= 16(0.2028) \rightarrow \text{solve} \\
0.5807a &= 3.2448 \rightarrow \text{solve} \\
\frac{\cancel{0.5807}a}{\cancel{0.5807}} &= \frac{3.2448}{0.5807} \\
a &\approx 5.6 \text{ units}
\end{aligned}$$

b) To determine the length of side d , the Law of Sines must be applied.

$$\begin{aligned}
\frac{e}{\sin E} &= \frac{d}{\sin D} \\
\frac{e}{\sin E} &= \frac{d}{\sin D} \rightarrow e = 214.9,\ \angle D = 39.7^\circ, \angle E = 41.3^\circ \\
\frac{214.9}{\sin(41.3^\circ)} &= \frac{d}{\sin(39.7^\circ)} \rightarrow \text{simplify} \\
214.9(\sin(39.7^\circ)) &= d(\sin(41.3^\circ)) \rightarrow \text{simplify} \\
214.9(0.6388) &= d(0.6600) \rightarrow \text{simplify} \\
137.2781 &= (0.6600)d \rightarrow \text{solve} \\
\frac{137.2781}{0.6600} &= \frac{\cancel{(0.6600)}}{\cancel{0.6600}}d \rightarrow \text{solve} \\
208.0 \text{ units} &\approx d
\end{aligned}$$

c) Cannot determine the length of side i . There is not enough information provided.

d) To determine the length of side l , the measure of $\angle K$ must be calculated first. This can be done by adding the two given angles and subtracting their sum from 180° . Then the Law of Sines can be used to determine the length of side l .

$$\begin{aligned}
\angle K &= 180^\circ - (16.2^\circ + 40.3^\circ) \\
\angle K &= 180^\circ - (56.5^\circ) \\
\angle K &= 123.5^\circ
\end{aligned}$$

$$\frac{k}{\sin K} = \frac{l}{\sin L}$$
$$\frac{k}{\sin K} = \frac{l}{\sin L} \rightarrow k = 6.3,\ \angle K = 123.5^\circ, \angle L = 40.3^\circ$$
$$\frac{6.3}{\sin(123.5^\circ)} = \frac{l}{\sin(40.3^\circ)} \rightarrow \text{simplify}$$
$$6.3(\sin(40.3^\circ)) = l(\sin(123.5^\circ)) \rightarrow \text{simplify}$$
$$6.3(0.6468) = (0.8339)l \rightarrow \text{simplify}$$
$$4.0748 = (0.8339)l \rightarrow \text{solve}$$
$$\frac{4.0748}{0.8339} = \frac{\cancel{(0.8339)}}{\cancel{0.8339}}l \rightarrow \text{solve}$$
$$4.9 \text{ units} \approx l$$

e) To determine the length of side o, the Law of Sines must be applied.

$$\frac{o}{\sin O} = \frac{m}{\sin M}$$
$$\frac{o}{\sin O} = \frac{m}{\sin M} \rightarrow m = 15,\ \angle O = 9^\circ, \angle M = 31^\circ$$
$$\frac{o}{\sin(9^\circ)} = \frac{15}{\sin(31^\circ)} \rightarrow \text{simplify}$$
$$o(\sin(31^\circ)) = 15(\sin(9^\circ)) \rightarrow \text{simplify}$$
$$o(0.5150) = 15(0.1564) \rightarrow \text{simplify}$$
$$(0.5150)o = 2.346 \rightarrow \text{solve}$$
$$\frac{\cancel{(0.5150)}}{\cancel{0.5150}}o = \frac{2.346}{0.5150} \rightarrow \text{solve}$$
$$o \approx 4.6 \text{ units}$$

f) To determine the length of side q , the Law of Sines must be applied.

$$\frac{q}{\sin Q} = \frac{r}{\sin R}$$
$$\frac{q}{\sin Q} = \frac{r}{\sin R} \rightarrow r = 3.62,\ \angle Q = 127^\circ,\ \angle R = 21.8^\circ$$
$$\frac{q}{\sin(127^\circ)} = \frac{3.62}{\sin(21.8^\circ)} \rightarrow \text{simplify}$$
$$q(\sin(21.8^\circ)) = 3.62(\sin(127^\circ)) \rightarrow \text{simplify}$$
$$q(0.3714) = 3.62(0.7986) \rightarrow \text{simplify}$$
$$(0.3714)q = 2.8909 \rightarrow \text{solve}$$
$$\frac{\cancel{(0.3714)}}{\cancel{0.3714}}q = \frac{2.8909}{0.3714} \rightarrow \text{solve}$$
$$q \approx 7.8 \text{ units}$$

4. To determine the length of side h , the Law of Sines may be used. The Law of Sines may also be used to determine the length of side g after the measure of $\angle G$ is calculated. This can be done by adding the two given angles and subtracting their sum from 180° .

$$\begin{aligned}
\angle G &= 180^\circ - (62.1^\circ + 21.3^\circ)\\
\angle G &= 180^\circ - (83.4^\circ)\\
\angle G &= 96.6^\circ
\end{aligned}$$

Side h

$$\begin{aligned}
\frac{h}{\sin H} &= \frac{i}{\sin I}\\
\frac{h}{\sin H} &= \frac{i}{\sin I} \rightarrow i = 108,\ \angle H = 62.1^\circ,\ \angle I = 21.3^\circ\\
\frac{h}{\sin(62.1^\circ)} &= \frac{108}{\sin(21.3^\circ)} \rightarrow \text{simplify}\\
h(\sin(21.3^\circ)) &= 108(\sin(62.1^\circ)) \rightarrow \text{simplify}\\
h(0.3633) &= 108(0.8838) \rightarrow \text{simplify}\\
(0.3633)h &= 95.450 \rightarrow \text{solve}\\
\frac{\cancel{(0.3633)}}{\cancel{0.3633}}h &= \frac{95.450}{0.3633} \rightarrow \text{solve}\\
h &\approx 262.7 \text{ units}
\end{aligned}$$

Side g

$$\begin{aligned}
\frac{g}{\sin G} &= \frac{i}{\sin I}\\
\frac{g}{\sin G} &= \frac{i}{\sin I} \rightarrow i = 108,\ \angle G = 96.6^\circ,\ \angle I = 21.3^\circ\\
\frac{g}{\sin(96.6^\circ)} &= \frac{108}{\sin(21.3^\circ)} \rightarrow \text{simplify}\\
g(\sin(21.3^\circ)) &= 108(\sin(96.6^\circ)) \rightarrow \text{simplify}\\
g(0.3633) &= 108(0.9934) \rightarrow \text{simplify}\\
(0.3633)g &= 107.287 \rightarrow \text{solve}\\
\frac{\cancel{(0.3633)}}{\cancel{0.3633}}g &= \frac{107.287}{0.3633} \rightarrow \text{solve}\\
g &\approx 295.3 \text{ units}
\end{aligned}$$

5.

$$\begin{aligned}
\frac{\sin A}{a} &= \frac{\sin B}{b}\\
b(\sin A) &= a(\sin B)\\
\frac{b(\sin A)}{b} &= \frac{a(\sin B)}{b}\\
\frac{\cancel{b}(\sin A)}{\cancel{b}} &= \frac{a(\sin B)}{b}\\
(\sin A) &= \frac{a(\sin B)}{b}\\
\frac{(\sin A)}{(\sin B)} &= \frac{a\cancel{(\sin B)}}{b\cancel{(\sin B)}}\\
\frac{(\sin A)}{(\sin B)} &= \frac{a}{b}
\end{aligned}$$

6. a) The Law of Cosines because the lengths of two sides and the included angle are given.

b) The triangle is a right triangle so the assumption that would be made is that one of the Trigonometric ratios would be used to determine the length of side x . However, there is not enough information given to conclude which ratio to apply.

c) Either the Law of Cosines or the Law of Sines could be used to calculate the measure of the angle.

d) The Law of Sines would be used to determine the length of side x .

7. a)

$$\begin{aligned}
\tan(54^\circ) &= \frac{\text{opp}}{\text{adj}}\\
\tan(54^\circ) &= \frac{x}{7.15}\\
1.3764 &= \frac{x}{7.15}\\
1.3764(7.15) &= (\cancel{7.15})\left(\frac{x}{\cancel{7.15}}\right)\\
9.84 \text{ units} &\approx x
\end{aligned}$$

$$\begin{aligned}
\tan(67^\circ) &= \frac{\text{opp}}{\text{adj}}\\
\tan(67^\circ) &= \frac{x}{9.84}\\
2.3559 &= \frac{x}{9.84}\\
2.3559(9.84) &= (\cancel{9.84})\left(\frac{x}{\cancel{9.84}}\right)\\
23.2 \text{ units} &\approx x
\end{aligned}$$

b) To determine the length of side x , the Law of Cosines can be used to determine the measure of the supplementary angle. This measurement can then be subtracted from 180° to calculate the measure of the corresponding angle and the Law of Sines can then be applied.

$$\begin{aligned}
\cos\angle A &= \frac{b^2+c^2-a^2}{2bc}\\
\cos\angle A &= \frac{b^2+c^2-a^2}{2bc} \rightarrow a = 11.2,\ b = 12.6,\ c = 8.9\\
\cos\angle A &= \frac{(12.6)^2+(8.9)^2-(11.2)^2}{2(12.6)(8.9)} \rightarrow \text{simplify}\\
\cos\angle A &= \frac{112.53}{224.28} \rightarrow \text{divide}\\
\cos\angle A &= 0.5017\\
\cos^{-1}(\cos\angle A) &= \cos^{-1}(0.5017)\\
\angle A &\approx 59.9^\circ
\end{aligned}$$

Supplementary Angle: $180^\circ - 59.9^\circ = 120.1^\circ$. This is also the corresponding angle in the other triangle.

$$
\begin{aligned}
\frac{a}{\sin A} &= \frac{c}{\sin C}\\
\frac{a}{\sin A} &= \frac{c}{\sin C} \rightarrow c = 8.9,\ \angle A = 120.1^\circ,\ \angle C = 31^\circ\\
\frac{a}{\sin(120.1^\circ)} &= \frac{8.9}{\sin(31^\circ)} \rightarrow \text{simplify}\\
a(\sin(31^\circ)) &= 8.9(\sin(120.1^\circ)) \rightarrow \text{simplify}\\
a(0.5150) &= 8.9(0.8652) \rightarrow \text{simplify}\\
(0.5150)a &= 7.7 \rightarrow \text{solve}\\
\frac{\cancel{(0.5150)}}{\cancel{0.5150}}a &= \frac{7.7}{0.5150} \rightarrow \text{solve}\\
a &\approx 15.0 \text{ units}
\end{aligned}
$$

8. There is not enough information given to complete this problem.

9. To determine the time that the driver must leave the warehouse, the total distance she travels and the length of time to travel the distance must be calculated. The distance between Stop B and Stop C can be determined by using the Law of Sines. The distance between Stop A and Stop C can also be determined by using the Law of Sines. The angle formed by the intersection of Stop C and Route 52 can be calculated by subtracting the sum of the other 2 angles from 180° .

$$
\begin{aligned}
\angle C &= 180^\circ - (41^\circ + 103^\circ)\\
\angle C &= 180^\circ - (144^\circ)\\
\angle C &= 36^\circ
\end{aligned}
$$

Distance between Stop B and Stop C (a)

$$
\begin{aligned}
\frac{a}{\sin A} &= \frac{c}{\sin C}\\
\frac{a}{\sin A} &= \frac{c}{\sin C} \rightarrow c = 12.3,\ \angle A = 41^\circ,\ \angle C = 36^\circ\\
\frac{a}{\sin(41^\circ)} &= \frac{12.3}{\sin(36^\circ)} \rightarrow \text{simplify}\\
a(\sin(36^\circ)) &= 12.3(\sin(41^\circ)) \rightarrow \text{simplify}\\
a(0.5878) &= 12.3(0.6561) \rightarrow \text{simplify}\\
\frac{\cancel{(0.5878)}}{\cancel{0.5878}}a &= \frac{8.070}{0.5878} \rightarrow \text{solve}\\
a &\approx 13.8 \text{ units}
\end{aligned}
$$

Distance between Stop A and Stop C (b)

$$\begin{aligned}
\frac{b}{\sin B} &= \frac{c}{\sin C} \\
\frac{b}{\sin B} &= \frac{c}{\sin C} \rightarrow c = 12.3,\ \angle B = 103^\circ,\ \angle C = 36^\circ \\
\frac{b}{\sin(103^\circ)} &= \frac{12.3}{\sin(36^\circ)} \rightarrow \text{simplify} \\
b(\sin(36^\circ)) &= 12.3(\sin(103^\circ)) \rightarrow \text{simplify} \\
b(0.5878) &= 12.3(0.9744) \rightarrow \text{simplify} \\
b(0.5878) &= 11.985 \rightarrow \text{solve} \\
\frac{\cancel{(0.5878)}}{\cancel{0.5878}} b &= \frac{11.985}{0.5878} \rightarrow \text{solve} \\
b &\approx 20.4 \text{ miles}
\end{aligned}$$

The total distance the driver must travel is $1.1 + 12.3 + 20.4 + 13.8 + 1.1 = 48.7$ miles

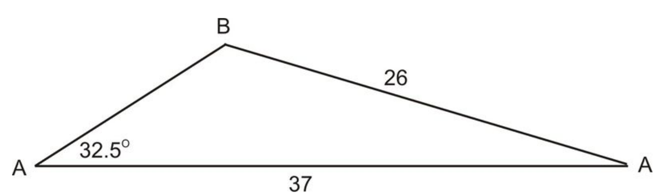
To travel this distance at a speed of 45 mph will take the driver $\frac{48.7 \text{ miles}}{45 \text{ mph}} \approx 1.1$ hours or 1 hour and 6 minutes . The driver must add to this time, the time needed to deliver each package. Now the total time is 1 hour 12 minutes . In order to return to the warehouse by 10:00 a.m., she must leave the warehouse at 8:48 a.m.

10. The information given in his problem is not sufficient to obtain an answer. If an angle of elevation increases, then the observer must be closer to the object. If this is the case, then the problem does not work.

The Ambiguous Case

Review Exercises:

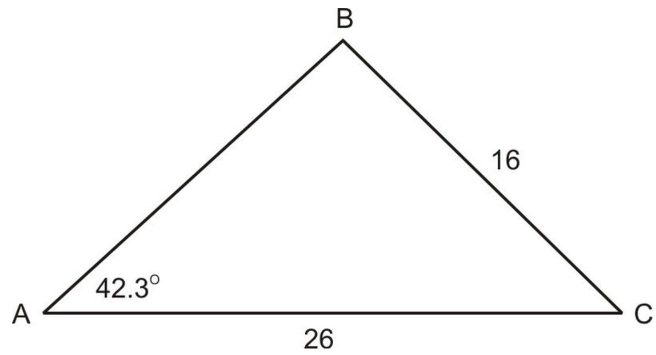
1. a)



$$\begin{aligned}
&b \sin A \\
&b \sin A \rightarrow b = 37,\ \angle A = 32.5^\circ \\
&(37)\sin(32.5^\circ) \rightarrow \text{simplify} \\
&(37)(0.5373) \rightarrow \text{simplify} \\
&(37)(0.5373) \approx 19.9
\end{aligned}$$

Therefore $a > b\ \sin A$ and there will be two solutions.

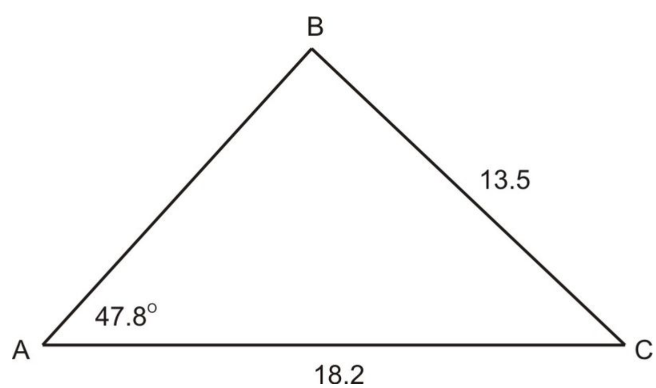
b)



$$
\begin{aligned}
&b \sin A\\
&b \sin A \to b = 26,\ \angle A = 42.3^\circ\\
&(26)\sin(42.3^\circ) \to \text{simplify}\\
&(26)(0.6730) \to \text{simplify}\\
&(26)(0.6730) \approx 17.5
\end{aligned}
$$

Therefore $a < b \sin A$ and there are no solutions.

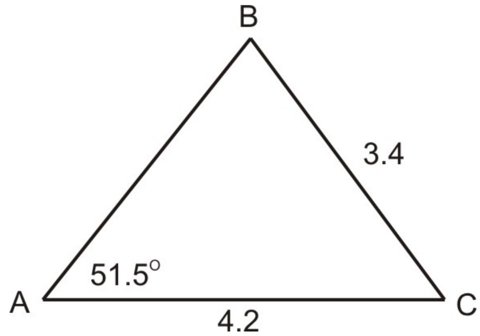
c)



$$
\begin{aligned}
&b \sin A\\
&b \sin A \to b = 18.2,\ \angle A = 47.8^\circ\\
&(18.2)\sin(47.8^\circ) \to \text{simplify}\\
&(18.2)(0.7408) \to \text{simplify}\\
&(18.2)(0.7408) \approx 13.5
\end{aligned}
$$

Therefore $a = b\ \sin A$ and there is one solution.

d)



$$\begin{aligned}
&b \sin A\\
&b \sin A \rightarrow b = 4.2,\ \angle A = 51.5^\circ\\
&(4.2) \sin(51.5^\circ) \rightarrow \text{simplify}\\
&(4.2)(0.7826) \rightarrow \text{simplify}\\
&(4.2)(0.7826) \approx 3.3
\end{aligned}$$

Therefore $a > b\ \sin A$ and there will be two solutions.

2. a)

$$\begin{aligned}
\frac{\sin A}{a} &= \frac{\sin B}{b}\\
\frac{\sin A}{a} &= \frac{\sin B}{b} \rightarrow \angle A = 32.5^\circ,\ a = 26,\ b = 37\\
\frac{\sin(32.5^\circ)}{26} &= \frac{\sin B}{37} \rightarrow \text{simplify}\\
\sin(32.5^\circ)(37) &= (26) \sin B \rightarrow \text{simplify}\\
(0.5373)(37) &= (26) \sin B \rightarrow \text{simplify}\\
19.8801 &= (26) \sin B \rightarrow \text{solve}\\
\frac{19.8801}{26} &= \frac{\cancel{(26)} \sin B}{\cancel{26}} \rightarrow \text{solve}\\
0.7646 &= \sin B \rightarrow \text{solve}\\
\sin^{-1}(0.7646) &= \sin^{-1}(\sin B)\\
49.9^\circ &\approx \angle B\\
\text{OR}\ \ \angle B &= 180^\circ - 49.9^\circ = 130.1^\circ
\end{aligned}$$

b) There is no solution as proven above in question 1.

c)

$$
\begin{aligned}
\frac{\sin A}{a} &= \frac{\sin B}{b}\\
\frac{\sin A}{a} &= \frac{\sin B}{b} \to \angle A = 47.8^\circ,\ a = 13.5,\ b = 18.2\\
\frac{\sin(47.8^\circ)}{13.5} &= \frac{\sin B}{18.2} \to \text{simplify}\\
\sin(47.8^\circ)(18.2) &= (13.5)\sin B \to \text{simplify}\\
(0.7408)(18.2) &= (13.5)\sin B \to \text{simplify}\\
13.4826 &= (13.5)\sin B \to \text{solve}\\
\frac{13.4826}{13.5} &= \frac{\cancel{(13.5)}\sin B}{\cancel{13.5}} \to \text{solve}\\
0.9987 &= \sin B \to \text{solve}\\
\sin^{-1}(0.9987) &= \sin^{-1}(\sin B)\\
87.1^\circ &\approx \angle B
\end{aligned}
$$

d)

$$
\begin{aligned}
\frac{\sin A}{a} &= \frac{\sin B}{b}\\
\frac{\sin A}{a} &= \frac{\sin B}{b} \to \angle A = 51.5^\circ,\ a = 3.4,\ b = 4.2\\
\frac{\sin(51.5^\circ)}{3.4} &= \frac{\sin B}{4.2} \to \text{simplify}\\
(4.2)\sin(51.5^\circ) &= (3.4)\sin B \to \text{simplify}\\
(4.2)(0.7826) &= (3.4)\sin B \to \text{simplify}\\
3.2869 &= (3.4)\sin B \to \text{solve}\\
\frac{3.2869}{3.4} &= \frac{\cancel{(3.4)}\sin B}{\cancel{3.4}} \to \text{solve}\\
0.9667 &= \sin B\\
\sin^{-1}(0.9667) &= \sin^{-1}(\sin B)\\
75.2^\circ &\approx \angle B\\
\text{OR } \angle B &= 180^\circ - 75.2^\circ = 104.8^\circ
\end{aligned}
$$

3.

$$
\begin{aligned}
\frac{\sin A}{a} &= \frac{\sin C}{c} \\
(ac)\left(\frac{\sin A}{a}\right) &= (ac)\left(\frac{\sin C}{c}\right) \rightarrow \text{simplify} \\
(\cancel{a}c)\left(\frac{\sin A}{\cancel{a}}\right) &= (a\cancel{c})\left(\frac{\sin C}{\cancel{c}}\right) \rightarrow \text{simplify} \\
(c)(\sin A) &= (a)(\sin C) \rightarrow \text{simplify} \\
c\text{Sin}A - c\text{Sin}C &= a\text{Sin}C - c\text{Sin}C \rightarrow \text{common factor} \\
(c)(\text{Sin}A - \text{Sin}C) &= (\text{Sin}C)(a-c) \rightarrow \text{divide}(c\text{Sin}C) \\
\frac{(c)(\text{Sin}A - \text{Sin}C)}{c\text{Sin}C} &= \frac{(\text{Sin }C)(a-c)}{c\text{Sin}C} \rightarrow \text{simplify} \\
\frac{(\cancel{c})(\text{Sin}A - \text{Sin}C)}{\cancel{c}\text{Sin}C} &= \frac{(\cancel{\text{Sin}C})(a-c)}{c\,\cancel{\text{Sin}C}} \rightarrow \text{simplify} \\
\frac{\text{Sin}A - \text{Sin}C}{\text{Sin}C} &= \frac{(a-c)}{c}
\end{aligned}
$$

4. Given $\triangle ABC \rightarrow a = 30 \text{ cm},\ c = 42 \text{ cm},\ \angle A = 38^\circ$. To calculate the measure of $\angle C$, the Law of Sines must be used.

$$
\begin{aligned}
\frac{\sin A}{a} &= \frac{\sin C}{c} \\
\frac{\sin A}{a} &= \frac{\sin C}{c} \rightarrow a = 30 \text{ cm},\ c = 42 \text{ cm},\ \angle A = 38^\circ \\
\frac{\sin(38^\circ)}{30} &= \frac{\sin C}{42} \rightarrow \text{simplify} \\
\frac{0.6157}{30} &= \frac{\sin C}{42} \rightarrow \text{simplify} \\
(0.6157)(42) &= (30)(\sin C) \rightarrow \text{simplify} \\
25.8594 &= 30\sin C \rightarrow \text{simplify} \\
\frac{25.8594}{30} &= \frac{\cancel{30}\sin C}{\cancel{30}} \rightarrow \text{simplify} \\
0.8620 &= \sin C \\
\sin^{-1}(0.8620) &= \sin^{-1}(\sin C) \\
59.5^\circ &\approx \angle C \\
\text{OR } \angle C &= 180^\circ - 59.5^\circ = 120.5^\circ
\end{aligned}
$$

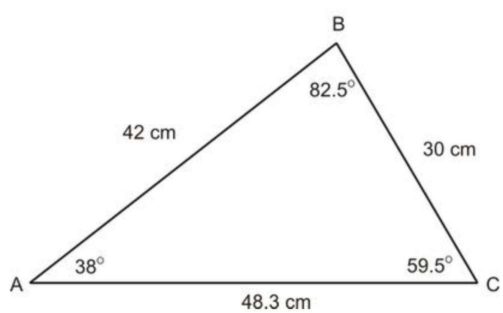
There are two solutions which results in two triangles. The sine function is positive in both the 1st and 2nd quadrant. The two possibilities are given above and both will satisfy the measure of the angle. Therefore, the length of side [U+0080][U+0098]b[U+0080][U+0099] will depend upon its corresponding angle.

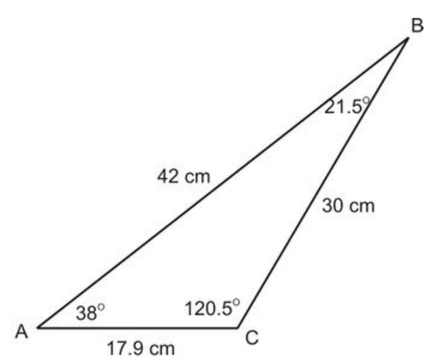
$$
\begin{aligned}
\angle B &= 180^\circ - (38^\circ + 59.5^\circ) \\
\angle B &= 180^\circ - (97.5^\circ) \\
\angle B &= 82.5^\circ
\end{aligned}
\qquad \text{OR} \qquad
\begin{aligned}
\angle B &= 180^\circ - (38^\circ + 120.5^\circ) \\
\angle B &= 180^\circ - (158.5^\circ) \\
\angle B &= 21.5^\circ
\end{aligned}
$$

$$\begin{aligned}
\frac{\sin A}{a} &= \frac{\sin B}{b} \\
\frac{\sin A}{a} &= \frac{\sin B}{b} \rightarrow a = 30 \text{ cm}, \ \angle B = 82.5^\circ, \ \angle A = 38^\circ \\
\frac{\sin(38^\circ)}{30} &= \frac{\sin(82.5^\circ)}{b} \rightarrow \text{simplify} \\
\frac{0.6157}{30} &= \frac{0.9914}{b} \rightarrow \text{simplify} \\
(0.6157)(b) &= (30)(0.9914) \rightarrow \text{simplify} \\
0.6157b &= 29.742 \rightarrow \text{solve} \\
\frac{0.6157b}{0.6157} &= \frac{29.742}{0.6157} \rightarrow \text{solve} \\
b &\approx 48.3 \text{ cm}
\end{aligned}$$

$$\begin{aligned}
\frac{\sin A}{a} &= \frac{\sin B}{b} \\
\frac{\sin A}{a} &= \frac{\sin B}{b} \rightarrow a = 30 \text{ cm}, \ \angle B = 21.5^\circ, \ \angle A = 38^\circ \\
\frac{\sin(38^\circ)}{30} &= \frac{\sin(21.5^\circ)}{b} \rightarrow \text{simplify} \\
\frac{0.6157}{30} &= \frac{0.3665}{b} \rightarrow \text{simplify} \\
(0.6157)(b) &= (30)(0.3665) \rightarrow \text{simplify} \\
0.6157b &= 10.995 \rightarrow \text{solve} \\
\frac{0.6157b}{0.6157} &= \frac{10.995}{0.6157} \rightarrow \text{solve} \\
b &\approx 17.9 \text{ cm}
\end{aligned}$$

Two triangles exist:





5. If there is one solution, $a = b \ \sin A$. In order for this to be true, the measure of $\angle A$ must be calculated.

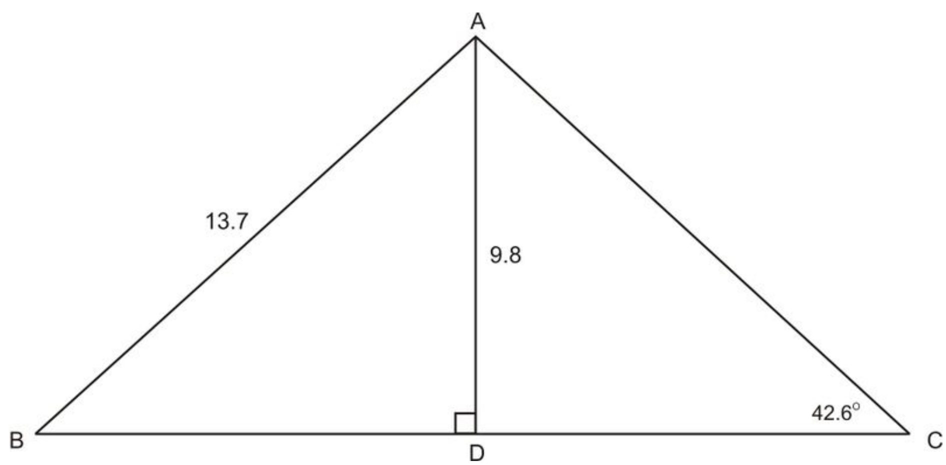
$$\begin{aligned}
a &= b\sin A \\
a &= b\sin A \rightarrow a = 22,\ b = 31 \\
22 &= 31\sin A \rightarrow \text{solve} \\
\frac{22}{31} &= \frac{\cancel{31}\sin A}{\cancel{31}} \rightarrow \text{solve} \\
0.7097 &= \sin A \\
\sin^{-1}(0.7097) &= \sin^{-1}(\sin A) \\
45.2^\circ &\approx \angle A
\end{aligned}$$

a) No solution means that $a = b\sin A$. This will occur when $\angle A$ is greater than 45.2° .

b) One solution means that $a = b\sin A$. This will occur when $\angle A$ equals 45.2° .

c) Two solutions mean that $a = b\sin A$. This will occur when $\angle A$ is less than 45.2° .

6. In the following triangle, the trigonometric ratios may be used to determine the measure of the required angles and sides or these may be used in conjunction with the Law of Cosines or the Law of Sines.



$$\begin{aligned}
&\triangle ACD \\
\sin\angle C &= \frac{\text{opp}}{\text{hyp}} \\
\sin(42.6^\circ) &= \frac{9.8}{x} \\
0.6769 &= \frac{9.8}{x} \\
0.6769(x) &= (\cancel{x})\left(\frac{9.8}{\cancel{x}}\right) \\
0.6769x &= 9.8 \\
\frac{0.6769}{0.6769}x &= \frac{9.8}{0.6769} \\
\frac{\cancel{0.6769}}{\cancel{0.6769}}x &= \frac{9.8}{0.6769} \\
x &\approx 14.5 \text{ units}
\end{aligned}$$

$$\begin{aligned}
&\triangle ABD \\
\sin\angle B &= \frac{\text{opp}}{\text{hyp}} \\
\sin\angle B &= \frac{9.8}{13.7} \\
\sin\angle B &= 0.7153 \\
\sin^{-1}(\sin\angle B) &= \sin^{-1}(0.7153) \\
\angle B &\approx 45.7^\circ
\end{aligned}$$

$$
\begin{aligned}
\angle A &= 180^\circ - (42.6^\circ + 45.7^\circ) \\
\angle A &= 180^\circ - (88.3^\circ) \\
\angle A &= 91.7^\circ
\end{aligned}
$$

$$
\begin{aligned}
&\triangle ABC \\
a^2 &= b^2 + c^2 - 2bc\cos A \\
a^2 &= b^2 + c^2 - 2bc\cos A \to b = 14.5,\ c = 13.7,\ \angle A = 91.7^\circ \\
a^2 &= (14.5)^2 + (13.7)^2 - 2(14.5)(13.7)\cos(91.7^\circ) \to \text{simplify} \\
a^2 &= 409.7264\ \to\ \sqrt{\text{both sides}} \\
\sqrt{a^2} &= \sqrt{409.7264} \to \text{simplify} \\
a &\approx 20.2 \text{ units}
\end{aligned}
$$

The required measurements are: $\angle A = 91.7^\circ,\ \angle B = 45.7^\circ,\ AC = 14.5 \text{ units},\ Bc = 20.2 \text{ units}$

7. To determine the measurements of the required sides and angles, the Law of Cosines, the Law of Sines, supplementary angles and the sum of the angles of a triangle must be used.

Begin with $\triangle BED$ since the length of each side is given. Use the Law of Cosines to determine the measure of the angles and then apply the sum of the angles in a triangle to determine the third angle. Then continue until the measure of each angle has been calculated.

$$
\begin{aligned}
&\triangle BED \\
\cos B &= \frac{d^2 + e^2 - b^2}{2de} \\
\cos B &= \frac{d^2 + e^2 - b^2}{2de} \to b = 7.6,\ d = 9.9,\ e = 10.2 \\
\cos B &= \frac{(9.9)^2 + (10.2)^2 - (7.2)^2}{2(9.9)(10.2)} \to \text{simplify} \\
\cos B &= \frac{144.29}{201.96} \to \text{divide} \\
\cos B &= 0.7144 \\
\cos^{-1}(\cos B) &= \cos^{-1}(0.7144) \\
\angle B &\approx 44.4^\circ
\end{aligned}
$$

$\angle D$ and $\angle BDC$ are supplementary angles

$\therefore\ \angle BDC = 180^\circ - 65.7^\circ = 114.3^\circ$

$$\cos E = \frac{b^2+d^2-e^2}{2bd}$$
$$\cos E = \frac{b^2+d^2-e^2}{2bd} \rightarrow b=7.6,\ d=9.9,\ e=10.2$$
$$\cos E = \frac{(7.6)^2+(9.9)^2-(10.2)^2}{2(7.6)(9.9)} \rightarrow \text{simplify}$$
$$\cos E = \frac{51.73}{150.48} \rightarrow \text{divide}$$
$$\cos E = 0.3438$$
$$\cos^{-1}(\cos E) = \cos^{-1}(0.3438)$$
$$\angle E \approx 69.9^\circ$$

$\angle E$ and $\angle BEA$ are supplementary angles

$\therefore \angle BEA = 180^\circ - 69.9^\circ = 110.1^\circ$

$$\angle D = 180^\circ - (44.4^\circ + 69.9^\circ)$$
$$\angle D = 180^\circ - (114.3^\circ)$$
$$\angle D = 65.7^\circ$$

In $\triangle CBD$

$\angle B = 180^\circ - (114.3^\circ + 21.8^\circ)$

$\angle B = 180^\circ - (136.1^\circ)$

$\angle B = 43.9^\circ$

In $\triangle ABC$

$\angle A = 180^\circ - (109.6^\circ + 21.8^\circ)$

$\angle A = 180^\circ - (131.4^\circ)$

$\angle A = 48.6^\circ$

In $\triangle ABE$

$\angle B = 180^\circ - (110.1^\circ + 48.6^\circ)$

$\angle B = 180^\circ - (158.7^\circ)$

$\angle B = 21.3^\circ$

In $\triangle ABE$, the length of AB is determined by using the Law of Sines.

$$\frac{a}{\sin A} = \frac{e}{\sin E}$$
$$\frac{a}{\sin A} = \frac{e}{\sin E} \rightarrow a = 9.9,\ \angle A = 48.6^\circ,\ \angle E = 110.1^\circ$$
$$\frac{9.9}{\sin(48.6^\circ)} = \frac{e}{\sin(110.1^\circ)} \rightarrow \text{simplify}$$
$$(9.9)(\sin(110.1^\circ)) = (\sin(48.6^\circ))e \rightarrow \text{simplify}$$
$$9.9(0.9391) = 0.7501e \rightarrow \text{simplify}$$
$$9.2971 = 0.7501e \rightarrow \text{solve}$$
$$\frac{9.2971}{0.7501} = \frac{\cancel{0.7501}e}{\cancel{0.7501}} \rightarrow \text{solve}$$
$$12.4 \text{ units} \approx e(AB)$$

In $\triangle BCD$, the length of BC is determined by using the Law of Sines.

$$\begin{aligned}
\frac{c}{\sin C} &= \frac{d}{\sin D} \\
\frac{c}{\sin C} &= \frac{d}{\sin D} \rightarrow c = 10.2,\ \angle C = 21.8^\circ,\ \angle D = 114.3^\circ \\
\frac{10.2}{\sin(21.8^\circ)} &= \frac{d}{\sin(114.3^\circ)} \rightarrow \text{simplify} \\
(\sin(114.3^\circ))10.2 &= (\sin(21.8^\circ))d \rightarrow \text{simplify} \\
(0.9114)10.2 &= (0.3714)d \rightarrow \text{simplify} \\
9.2963 &= (0.3714)d \rightarrow \text{solve} \\
\frac{9.2963}{0.3714} &= \frac{\cancel{(0.3714)}d}{\cancel{0.3714}} \rightarrow \text{solve} \\
25.0 \text{ units} &\approx d(BC)
\end{aligned}$$

In $\triangle BCD$, the length of DC is determined by using the Law of Sines

$$\begin{aligned}
\frac{c}{\sin C} &= \frac{b}{\sin B} \\
\frac{c}{\sin C} &= \frac{b}{\sin B} \rightarrow c = 10.2,\ \angle C = 21.8^\circ,\ \angle B = 43.9^\circ \\
\frac{10.2}{\sin(21.8^\circ)} &= \frac{b}{\sin(43.9^\circ)} \rightarrow \text{simplify} \\
(\sin(43.9^\circ))10.2 &= (\sin(21.8^\circ))b \rightarrow \text{simplify} \\
(0.6934)10.2 &= (0.3714)b \rightarrow \text{simplify} \\
7.0727 &= (0.3714)b \rightarrow \text{solve} \\
\frac{7.0727}{0.3714} &= \frac{\cancel{(0.3714)}b}{\cancel{0.3714}} \rightarrow \text{solve} \\
19.0 \text{ units} &\approx b(CD)
\end{aligned}$$

In $\triangle ABC$, the Law of Cosines may be used to calculate the length of side $b\ (AC)$

$$\begin{aligned}
b^2 &= a^2 + c^2 - 2ac\ \cos B \\
b^2 &= a^2 + c^2 - 2ac\ \cos B \rightarrow a = 25,\ c = 12.4,\ \angle B = 109.6^\circ \\
b^2 &= (25)^2 + (12.4)^2 - 2(25)(12.4)\cos(109.6^\circ) \rightarrow \text{simplify} \\
b^2 &= 986.7400\ \rightarrow\ \sqrt{\text{both sides}} \\
\sqrt{b^2} &= \sqrt{986.7400} \rightarrow \text{simplify} \\
b(AC) &\approx 31.4 \text{ units}
\end{aligned}$$

In $\triangle ABE$, the length of AE is the difference between the length of AC and the sum of ED and CD .

$$\begin{aligned}
AE &= AC - (ED + CD) \\
AE &= 31.4 - (7.6 + 19.0) \\
AE &= 4.8 \text{ units}
\end{aligned}$$

The solutions are:

a) $BC = 25.0$ units

b) $AB = 12.4$ units

c) $AC = 31.4$ units

d) $AE = 4.8$ units

e) $ED = 7.6$ units (This was given)

f) $DC = 19.0$ units

g) $\angle ABE = 21.3^\circ$

h) $\angle BEA = 110.1^\circ$

i) $\angle BAE = 48.6^\circ$

j) $\angle BED = 69.9^\circ$

k) $\angle EDB = 65.7^\circ$

l) $\angle DBE = 44.4^\circ$

m) $\angle DBC = 43.9^\circ$

n) $\angle BDC = 114.3^\circ$

8. Let $S_1 = A,\ S_2 = B,\ S_3 = C$. The Law of Sines may be used to determine the measure of $\angle B$ and then either the Law of Sines or the Law of Cosines may be used to determine the length of side a .

$$
\begin{aligned}
\frac{c}{\sin C} &= \frac{b}{\sin B} \\
\frac{c}{\sin C} &= \frac{b}{\sin B} \rightarrow c = 4500,\ \angle C = 56^\circ,\ \angle b = 4000^\circ \\
\frac{4500}{\sin(56^\circ)} &= \frac{4000}{\sin B} \rightarrow \text{simplify} \\
4500(\sin(B)) &= 4000(\sin(56^\circ)) \rightarrow \text{simplify} \\
4500 \sin B &= 4000(0.8290) \rightarrow \text{simplify} \\
4500 \sin B &= 3316.1503 \rightarrow \text{solve} \\
\frac{\cancel{4500} \sin B}{\cancel{4500}} &= \frac{3316.1503}{4500} \rightarrow \text{solve} \\
\sin B &= 0.7369 \\
\sin^{-1}(\sin B) &= \sin^{-1}(0.7369) \\
\angle B &\approx 47.5^\circ
\end{aligned}
$$

$$
\begin{aligned}
\angle A &= 180^\circ - (56^\circ + 47.5^\circ) \\
\angle A &= 180^\circ - (103.5^\circ) \\
\angle A &= 76.5^\circ
\end{aligned}
$$

$$a^2 = b^2 + c^2 - 2bc\cos A$$
$$a^2 = b^2 + c^2 - 2bc\cos A \rightarrow b = 4000,\ c = 4500,\ angle A = 76.5^\circ$$
$$a^2 = (4000)^2 + (4500)^2 - 2(4000)(4500)\cos(76.5^\circ) \rightarrow \text{simplify}$$
$$a^2 = 27845966.9 \ \rightarrow \sqrt{\text{both sides}}$$
$$\sqrt{a^2} = \sqrt{27845966.9} \rightarrow \text{simplify}$$
$$a \approx 5276.9 \text{ ft}$$

The distance between Sensor 3 and Sensor 2 is approximately 5276.9 feet . If the range of Sensor 3 is 6000 feet , it will be able to detect all movement from its location to Sensor 2.

9. Let $S_4 = D$.

$$\angle D = 180^\circ - (36^\circ + 49^\circ)$$
$$\angle D = 180^\circ - (85^\circ)$$
$$\angle D = 95^\circ$$

The Law of Sines may be used to determine the distance between Sensor 2 and Sensor 4, as well as the distance between Sensor 3 and Sensor 4.

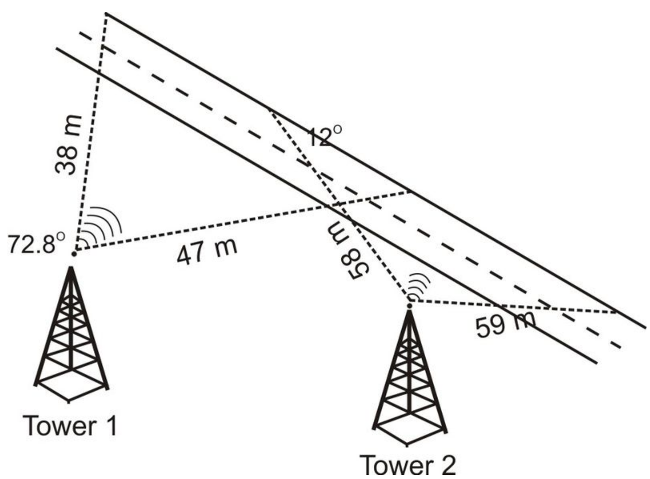
$$\frac{c}{\sin C} = \frac{d}{\sin D}$$
$$\frac{c}{\sin C} = \frac{d}{\sin D} \rightarrow \angle c = 49^\circ,\ \angle D = 95^\circ,\ \angle d = 5276.9$$
$$\frac{c}{\sin(49^\circ)} = \frac{5276.9}{\sin(95^\circ)} \rightarrow \text{simplify}$$
$$(\sin(95^\circ))c = 5276.9(\sin(49^\circ)) \rightarrow \text{simplify}$$
$$0.9962c = 5276.9(0.7547) \rightarrow \text{simplify}$$
$$0.9962c = 3982.4764 \rightarrow \text{solve}$$
$$\frac{\cancel{0.9962}c}{\cancel{0.9962}} = \frac{3982.4764}{0.9962} \rightarrow \text{solve}$$
$$c \approx 3997.7 \text{ feet}$$

The distance between Sensor 2 and Sensor 4 is approximately 3997.7 feet .

$$\frac{b}{\sin B} = \frac{d}{\sin D}$$
$$\frac{b}{\sin B} = \frac{d}{\sin D} \to \angle B = 36^\circ,\ \angle D = 95^\circ,\ d = 5276.9$$
$$\frac{b}{\sin(36^\circ)} = \frac{5276.9}{\sin(95^\circ)} \to \text{simplify}$$
$$(\sin(95^\circ))b = 5276.9(\sin(36^\circ)) \to \text{simplify}$$
$$(0.9962)b = 5276.9(0.5875) \to \text{simplify}$$
$$(0.9962)b = 3101.6840 \to \text{solve}$$
$$\frac{\cancel{(0.9962)}b}{\cancel{0.9962}} = \frac{3101.6840}{0.9962} \to \text{solve}$$
$$b \approx 3113.5 \text{ feet}$$

The distance between Sensor 3 and Sensor 4 is approximately 3113.5 feet .

10. Company A - the law of cosines may be used to determine the distance over which a driver has cell phone service.



Company A

$$a^2 = b^2 + c^2 - 2bc\cos A$$
$$a^2 = b^2 + c^2 - 2bc\cos A \to b = 47,\ c = 38,\ \angle A = 72.8^\circ$$
$$a^2 = (47)^2 + (38)^2 - 2(47)(38)\cos(72.8^\circ) \to \text{simplify}$$
$$a^2 = 2596.7308\ \to\ \sqrt{\text{both sides}}$$
$$\sqrt{a^2} = \sqrt{2596.7308} \to \text{simplify}$$
$$a \approx 51.0 \text{ miles}$$

Company B

$$\frac{b}{\sin B} = \frac{e}{\sin E}$$
$$\frac{b}{\sin B} = \frac{e}{\sin E} \rightarrow b = 59,\ e = 58,\ \angle B = 12^\circ$$
$$\frac{59}{\sin(12^\circ)} = \frac{58}{\sin E} \rightarrow \text{simplify}$$
$$59(\sin E) = 58(\sin(12^\circ)) \rightarrow \text{simplify}$$
$$59(\sin E) = 58(0.2079) \rightarrow \text{simplify}$$
$$59(\sin E) = 12.0589 \rightarrow \text{solve}$$
$$\frac{\cancel{59}(\sin E)}{\cancel{59}} = \frac{12.0589}{59} \rightarrow \text{solve}$$
$$\sin E = 0.2044 \rightarrow \text{solve}$$
$$\sin^{-1}(\sin E) = \sin^{-1}(0.2044)$$
$$\angle E \approx 11.8^\circ$$

$$\angle D = 180^\circ - (12^\circ + 11.8^\circ)$$
$$\angle D = 180^\circ - (23.8^\circ)$$
$$\angle D = 156.2^\circ$$

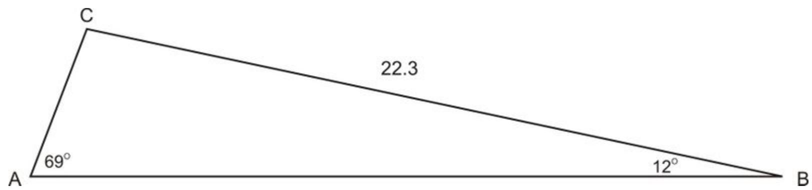
$$\frac{b}{\sin B} = \frac{d}{\sin D}$$
$$\frac{b}{\sin B} = \frac{d}{\sin D} \rightarrow b = 59,\ \angle B = 12^\circ,\ \angle D = 156.2^\circ$$
$$\frac{59}{\sin(12^\circ)} = \frac{d}{\sin(156.2^\circ)} \rightarrow \text{simplify}$$
$$59(\sin(156.2^\circ)) = (\sin(12^\circ))d \rightarrow \text{simplify}$$
$$59(0.4035) = (0.2079)d \rightarrow \text{simplify}$$
$$23.8092 = (0.2079)d \rightarrow \text{solve}$$
$$\frac{23.8092}{0.2079} = \frac{\cancel{0.2079}d}{\cancel{0.2079}} = \rightarrow \text{solve}$$
$$114.5 \text{ miles} \approx d$$

There is an overlap in cell phone service for approximately 63.5 miles .

General Solutions of Triangles

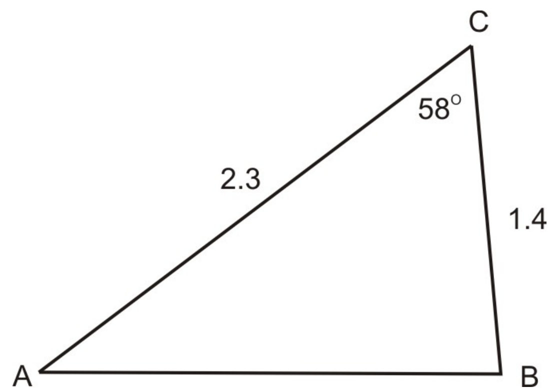
Review Exercises:

1. a) In the following triangle, the case AAS is given.



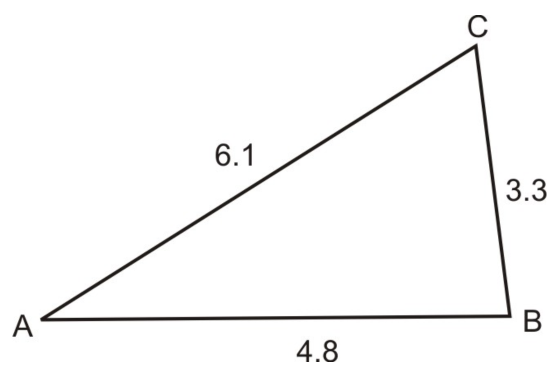
There is only one solution since the measure of two angles has been given. The Law of Sines would be used to determine the length of side b .

b) In the following triangle, the case SAS is given.



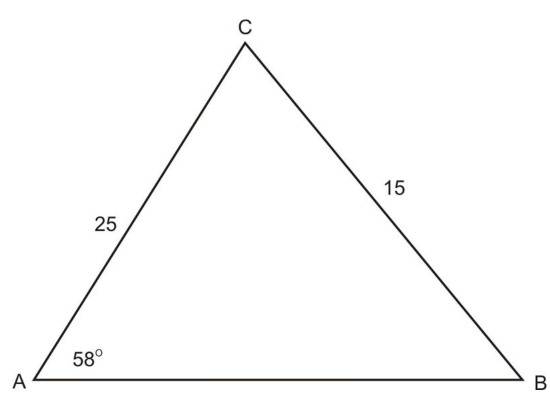
There is only one solution since the measure of two sides and the included angle has been given. The Law of Cosines would be used to determine the length of side c .

c) In the following triangle, the case SSS is given.



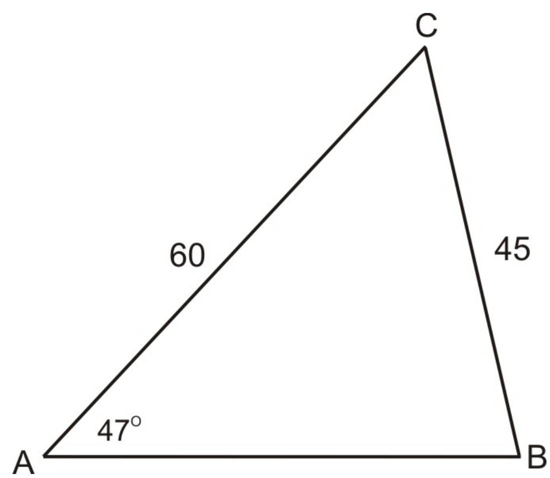
There is only one solution since the measure of the three sides has been given. The Law of Cosines would be used to determine the measure of $\angle A$.

d) In the following triangle, the case SSA is given.



The Law of Sines would be used to determine the measure of $\angle B$. However, when the Law of Sines is applied, there is no solution.

e) In the following triangle, the case SSA is given.



There are two solutions since the measure of one angle and the length of two sides has been given. The Law of Sines would be used to determine the measure of $\angle B$.

2. a)

$$\begin{aligned}
\frac{a}{\sin A} &= \frac{b}{\sin B} \\
\frac{a}{\sin A} &= \frac{b}{\sin B} \rightarrow a = 22.3,\ \angle A = 69^\circ,\ \angle B = 12^\circ \\
\frac{22.3}{\sin(69^\circ)} &= \frac{b}{\sin(12^\circ)} \rightarrow \text{simplify} \\
22.3(\sin(12^\circ)) &= (\sin(69^\circ))b \rightarrow \text{simplify} \\
22.3(0.2079) &= (0.9336)b \rightarrow \text{simplify} \\
4.6362 &= (0.9336)b \rightarrow \text{solve} \\
\frac{4.6362}{0.9336} &= \frac{\cancel{(0.9336)}b}{\cancel{0.9336}} \rightarrow \text{solve} \\
5.0 \text{ units} &\approx b
\end{aligned}$$

b)

$$\begin{aligned}
c^2 &= a^2 + b^2 - 2ab\cos C \\
c^2 &= a^2 + b^2 - 2ab\cos C \rightarrow a = 1.4,\ b = 2.3,\ \angle C = 58^\circ \\
c^2 &= (1.4)^2 + (2.3)^2 - 2(1.4)(2.3)\cos(58^\circ) \rightarrow \text{simplify} \\
c^2 &= 3.8373\ \rightarrow \sqrt{\text{both sides}} \\
\sqrt{c^2} &= \sqrt{3.8373} \rightarrow \text{simplify} \\
c &\approx 2.0 \text{ units}
\end{aligned}$$

c)

$$\begin{aligned}
\cos A &= \frac{b^2+c^2-a^2}{2bc}\\
\cos A &= \frac{b^2+c^2-a^2}{2bc} \ \rightarrow a=3.3,\ b=6.1,\ c=4.8\\
\cos A &= \frac{(6.1)^2+(4.8)^2-(3.3)^2}{2(6.1)(4.8)} \ \rightarrow \text{simplify}\\
\cos A &= \frac{49.36}{58.56} \rightarrow \text{divide}\\
\cos A &= 0.8429\\
\cos^{-1}(\cos A) &= \cos^{-1}(0.8429)\\
\angle A &\approx 32.6^\circ
\end{aligned}$$

d)

$$\begin{aligned}
\frac{a}{\sin A} &= \frac{b}{\sin B}\\
\frac{a}{\sin A} &= \frac{b}{\sin B} \rightarrow a=15,\ b=25,\ \angle A=58^\circ\\
\frac{15}{\sin(58^\circ)} &= \frac{25}{\sin B} \rightarrow \text{simplify}\\
15(\sin B) &= 25(\sin(58^\circ)) \rightarrow \text{simplify}\\
15(\sin B) &= 25(0.8480) \rightarrow \text{simplify}\\
15(\sin B) &= 21.2012 \rightarrow \text{solve}\\
\frac{\cancel{15}(\sin B)}{\cancel{15}} &= \frac{21.2012}{15} =\rightarrow \text{solve}\\
(\sin B) &= 1.4134\\
\sin^{-1}(\sin B) &= \sin^{-1}(1.4134)
\end{aligned}$$

Does Not Exist

e)

$$\begin{aligned}
\frac{a}{\sin A} &= \frac{b}{\sin B}\\
\frac{a}{\sin A} &= \frac{b}{\sin B} \rightarrow a=45,\ b=60, \angle A=47^\circ\\
\frac{45}{\sin(47^\circ)} &= \frac{60}{\sin B} \rightarrow \text{simplify}\\
45(\sin B) &= 60(\sin(47^\circ)) \rightarrow \text{simplify}\\
45(\sin B) &= 60(0.7314) \rightarrow \text{simplify}\\
45(\sin B) &= 43.884 \rightarrow \text{simplify}\\
\frac{\cancel{45}(\sin B)}{\cancel{45}} &= \frac{43.884}{45} =\rightarrow \text{solve}\\
(\sin B) &= 0.9752\\
\sin^{-1}(\sin B) &= \sin^{-1}(0.9752)\\
77.2^\circ &\approx \angle B\\
&\text{Or}\\
\angle B &= 180^\circ - 77.2^\circ = 102.8^\circ
\end{aligned}$$

3. The following information is still unknown:

a) c *and* $\angle C$

b) $\angle A$ *and* $\angle B$

c) $\angle B$ *and* $\angle C$

d) There is no solution

e) c *and* $\angle C$

4. When solving a triangle, a check list can be used to ensure that no parts have been missed.

$$\begin{aligned}\text{In } \triangle ABC \rightarrow a &= \_\_\_\_ \quad \angle A = \_\_\_\_ \\ b &= \_\_\_\_ \quad \angle B = \_\_\_\_ \\ c &= \_\_\_\_ \quad \angle C = \_\_\_\_\end{aligned}$$

a)

$$\begin{aligned}\angle C &= 180^\circ - (12^\circ + 69^\circ) \\ \angle C &= 180^\circ - (81^\circ) \\ \angle C &= 99^\circ\end{aligned}$$

$$\begin{aligned}c^2 &= a^2 + b^2 - 2ab\cos C \\ c^2 &= a^2 + b^2 - 2ab\cos C \rightarrow a = 22.3,\ b = 5.0,\ \angle C = 99^\circ \\ c^2 &= (22.3)^2 + (5.0)^2 - 2(22.3)(5.0)\cos(99^\circ) \rightarrow \text{simplify} \\ c^2 &= 557.1749 \ \rightarrow \ \sqrt{\text{both sides}} \\ \sqrt{c^2} &= \sqrt{557.1749} \rightarrow \text{simplify} \\ c &\approx 23.6 \text{ units}\end{aligned}$$

$$\begin{aligned}\text{In } \triangle ABC \rightarrow a &= 22.3 \quad \angle A = 69^\circ \\ b &= 5.0 \quad \angle B = 12^\circ \qquad\qquad \text{SOLVED} \\ c &= 23.6 \quad \angle C = 99^\circ\end{aligned}$$

c)

$$\begin{aligned}\cos A &= \frac{b^2 + c^2 - a^2}{2bc} \\ \cos A &= \frac{b^2 + c^2 - a^2}{2bc} \rightarrow a = 1.4,\ b = 2.3,\ c = 2.0 \\ \cos A &= \frac{(2.3)^2 + (2.0)^2 - (1.4)^2}{2(2.3)(2.0)} \rightarrow \text{simplify} \\ \cos A &= \frac{7.33}{9.2} \rightarrow \text{divide} \\ \cos A &= 0.7967 \\ \cos^{-1}(\cos A) &= \cos^{-1}(0.7967) \\ \angle A &\approx 37.2^\circ\end{aligned}$$

$$\angle B = 180^\circ - (58^\circ + 37.2^\circ)$$
$$\angle B = 180^\circ - (95.2^\circ)$$
$$\angle B = 84.8^\circ$$

$$\begin{aligned} \text{In } \triangle ABC \rightarrow a &= 1.4 \quad \angle A = 37.2^\circ \\ b &= 2.3 \quad \angle B = 84.8^\circ \qquad\qquad \text{SOLVED} \\ c &= 2.0 \quad \angle C = 58^\circ \end{aligned}$$

$$\begin{aligned} \cos B &= \frac{a^2 + c^2 - b^2}{2ac} \\ \cos B &= \frac{a^2 + c^2 - b^2}{2ac} \rightarrow a = 3.3,\ b = 6.1,\ c = 4.8 \\ \cos B &= \frac{(3.3)^2 + (4.8)^2 - (6.1)^2}{2(3.3)(4.8)} \rightarrow \text{simplify} \\ \cos A &= \frac{-3.28}{31.68} \rightarrow \text{divide} \\ \cos A &= -0.1035 \\ \cos^{-1}(\cos B) &= \cos^{-1}(-0.1035) \\ \angle B &\approx 95.9^\circ \end{aligned}$$

$$\angle A = 180^\circ - (95.9^\circ + 32.6^\circ)$$
$$\angle A = 180^\circ - (128.5^\circ)$$
$$\angle A = 51.5^\circ$$

$$\begin{aligned} \text{In } \triangle ABC \rightarrow a &= 3.3 \quad \angle A = 51.5^\circ \\ b &= 6.1 \quad \angle B = 95.9^\circ \qquad\qquad \text{SOLVED} \\ c &= 2.0 \quad \angle C = 32.6^\circ \end{aligned}$$

d) There is no solution.

e)

$$\begin{aligned} \angle C &= 180^\circ - (47^\circ + 77.2^\circ) & \text{OR} \qquad \angle C &= 180^\circ - (47^\circ + 102.8^\circ) \\ \angle C &= 180^\circ - (124.2^\circ) & \angle C &= 180^\circ - (149.8^\circ) \\ \angle C &= 55.8^\circ & \angle C &= 30.2^\circ \end{aligned}$$

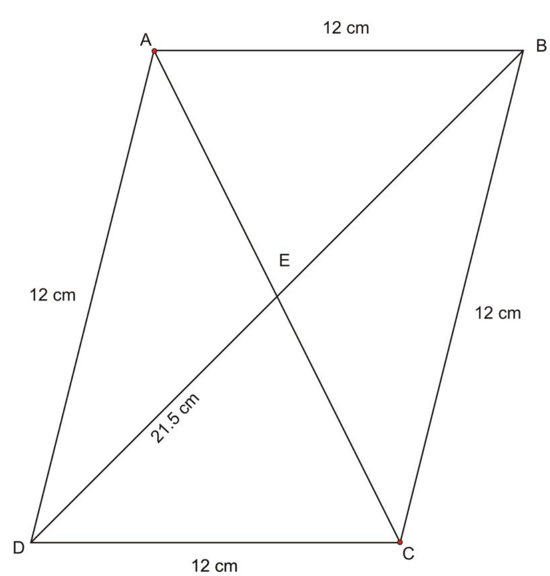
$$\begin{aligned}
c^2 &= a^2+b^2-2ab\cos C\\
c^2 &= a^2+b^2-2ab\cos C \rightarrow a=45,\ b=60,\ \angle C=55.8^\circ\\
c^2 &= (45)^2+(60)^2-2(45)(60)\cos(55.8^\circ) \rightarrow \text{simplify}\\
c^2 &= 2589.7498 \ \rightarrow\ \sqrt{\text{both sides}}\\
\sqrt{c^2} &= \sqrt{2589.7498} \rightarrow \text{simplify}\\
c &\approx 50.9 \text{ units}\\
c^2 &= a^2+b^2-2ab\cos C\\
c^2 &= a^2+b^2-2ab\cos C \rightarrow a=45,\ b=60,\ \angle C=30.2^\circ\\
c^2 &= (45)^2+(60)^2-2(45)(60)\cos(30.2^\circ) \rightarrow \text{simplify}\\
c^2 &= 957.9161 \rightarrow\ \sqrt{\text{both sides}}\\
\sqrt{c^2} &= \sqrt{957.9161} \rightarrow \text{simplify}\\
c &\approx 31.0 \text{ units}
\end{aligned}$$

$$\begin{aligned}
\text{In } \triangle ABC \rightarrow a &= 45 \quad \angle A = 47^\circ \qquad\qquad \text{SOLVED}\\
b &= 60 \quad \angle B = 77.2^\circ\\
c &= 50.9 \quad \angle C = 55.8^\circ
\end{aligned}$$

OR

$$\begin{aligned}
\text{In } \triangle ABC \rightarrow a &= 45 \quad \angle A = 47^\circ\\
b &= 60 \quad \angle B = 103.8^\circ \qquad\qquad \text{SOLVED}\\
c &= 31 \quad \angle C = 30.2^\circ
\end{aligned}$$

5. The area of a rhombus is readily found by using the formula $A=\frac{1}{2}xy$ where *x and y* are the diagonals of the rhombus. These diagonals intersect at right angles.



The length of the diagonal BD is 21.5 cm . and is bisected by the shorter diagonal AC . There are four right triangles within the rhombus. To determine the length of the shorter diagonal, the Pythagorean Theorem can be used. This distance can be doubled to obtain the length of AC .

$$\begin{aligned}
&\triangle BEC\\
&\triangle BEC \rightarrow BE = \frac{1}{2}(21.5)10.75,\ BC = 12(\text{hyp})
\end{aligned}$$

In $\triangle BEC$, the Pythagorean Theorem must be used to calculate the length of EC .

$$\begin{aligned}
(h)^2 &= (s_1)^2 + (s_2)^2\\
(12)^2 &= (10.75)^2 + (s_2)^2\\
(12)^2 &= (10.75)^2 + (s_2)^2\\
28.4375 &= (s_2)^2\\
\sqrt{28.4375} &= \sqrt{(s_2)^2}\\
5.3 \text{ cm} &\approx s
\end{aligned}$$

The length of AC is $2(5.3 \text{ cm}) = 10.6 \text{ cm}$

The area of the rhombus is:

$$\begin{aligned}
&A = \frac{1}{2}xy\\
&A = \frac{1}{2}xy \rightarrow x = BD(21.5 \text{ cm},\ y = AC(10.6 \text{ cm})\\
&A = \frac{1}{2}(21.5)(10.6) \rightarrow \text{solve}\\
&A \approx 113.95 \text{ cm}^2
\end{aligned}$$

To calculate the measure of the angles of the rhombus, use trigonometric ratios.

$$\begin{aligned}
&\sin\angle C = \frac{\text{opp}}{\text{hyp}} && \cos\angle B = \frac{\text{adj}}{\text{hyp}}\\
&\sin\angle C = \frac{10.75}{12} && \cos\angle B = \frac{10.75}{12}\\
&\sin\angle C = 0.8958 && \cos\angle B = 0.8958\\
&\sin^{-1}(\sin\angle C) = \sin^{-1}(0.8958) && \cos^{-1}(\cos\angle B) = \cos^{-1}(0.8958)\\
&\angle C \approx 63.6^\circ && \angle B \approx 26.4^\circ
\end{aligned}$$

$$\begin{aligned}
&\angle BCD = 2(63.6^\circ) = 127.2^\circ && \angle ABC = 2(26.4^\circ) = 52.8^\circ\\
&\angle BAD = 2(63.6^\circ) = 127.2^\circ && \angle ADC = 2(26.4^\circ) = 52.8^\circ
\end{aligned}$$

6. To begin the solution to this question, begin by dividing the pentagon into 3 triangles. One triangle has vertices $1,2,5$. The second triangle has vertices $2,4,5$. The third triangle has vertices $2,3,4$.

$$\begin{aligned}
&\triangle 125\\
c^2 &= a^2+b^2-2ab\cos C\\
c^2 &= a^2+b^2-2ab\cos C \rightarrow a=192,\ b=190.5,\ \angle C=81^\circ\\
c^2 &= (192)^2+(190.5)^2-2(192)(190.5)\cos(81^\circ) \rightarrow \text{simplify}\\
c^2 &= 61710.7560 \rightarrow \sqrt{\text{both sides}}\\
\sqrt{c^2} &= \sqrt{61710.7560} \rightarrow \text{simplify}\\
c &\approx 248.4 \text{ units}
\end{aligned}$$

$$\begin{aligned}
\cos B &= \frac{a^2+c^2-b^2}{2ac}\\
\cos B &= \frac{a^2+c^2-b^2}{2ac} \rightarrow a=192,\ b=190.5,\ c=248.4\\
\cos B &= \frac{(192)^2+(248.4)^2-(190.5)^2}{2(192)(248.4)} \rightarrow \text{simplify}\\
\cos B &= \frac{62276.31}{95385.6} \rightarrow \text{divide}\\
\cos B &= 0.6529\\
\cos^{-1}(\cos B) &= \cos^{-1}(0.6529)\\
\angle B(\angle 2) &\approx 49.2^\circ
\end{aligned}$$

$$\begin{aligned}
\angle A &= 180^\circ-(81^\circ+49.2^\circ)\\
\angle A &= 180^\circ-(130.2^\circ)\\
\angle A(\angle 5) &= 49.8^\circ
\end{aligned}$$

Area of Triangle 1:

$$\begin{aligned}
K &= \frac{1}{2}ab\sin C\\
K &= \frac{1}{2}ab\sin C \rightarrow a=192,\ b=190.5,\ \angle C=81^\circ\\
K &= \frac{1}{2}(192)(190.5)\sin(81^\circ) \rightarrow \text{simplify}\\
K &= \frac{1}{2}(192)(190.5)(0.9877)\\
K &= 18,062.8 \text{ square units}
\end{aligned}$$

$$\begin{aligned}
&\triangle 234\\
e^2 &= b^2+d^2-2bd\cos E\\
e^2 &= b^2+d^2-2bd\cos E \rightarrow b=146,\ d=173.8,\ \angle C=73^\circ\\
e^2 &= (146)^2+(173.8)^2-2(146)(173.8)\cos(73^\circ) \rightarrow \text{simplify}\\
e^2 &= 36684.6929 \rightarrow \sqrt{\text{both sides}}\\
\sqrt{e^2} &= \sqrt{36684.6929} \rightarrow \text{simplify}\\
e &\approx 191.5 \text{ units}
\end{aligned}$$

$$\begin{aligned}
\cos B &= \frac{d^2+e^2-b^2}{2de}\\
\cos B &= \frac{d^2+e^2-b^2}{2de} \rightarrow d=173.8,\ e=191.5,\ b=146\\
\cos B &= \frac{(173.8)^2+(191.5)^2-(146)^2}{2(173.8)(191.5)} \rightarrow \text{simplify}\\
\cos B &= \frac{45562.69}{66565.4} \rightarrow \text{divide}\\
\cos B &= 0.6845\\
\cos^{-1}(\cos B) &= \cos^{-1}(0.6845)\\
\angle B(\angle 2) &\approx 46.8^\circ
\end{aligned}$$

$$\begin{aligned}
\angle D &= 180^\circ-(73^\circ+46.8^\circ)\\
\angle D &= 180^\circ-(119.8^\circ)\\
\angle D(\angle 4) &= 60.2^\circ
\end{aligned}$$

Area of Triangle 3:

$$\begin{aligned}
K &= \frac{1}{2}bd\sin E\\
K &= \frac{1}{2}bd\sin E \rightarrow b=146,\ d=173.8,\ \angle E=73^\circ\\
K &= \frac{1}{2}(146)(173.8)\sin(73^\circ) \rightarrow \text{simplify}\\
K &= \frac{1}{2}(146)(173.8)(0.9563)\\
K &= 12,133.0 \text{ square units}
\end{aligned}$$

$$\begin{aligned}
&\triangle 245\\
\cos B &= \frac{a^2+d^2-b^2}{2ad}\\
\cos B &= \frac{a^2+d^2-b^2}{2ad} \ \rightarrow a = 191.5,\ d = 248.4,\ e = 118\\
\cos B &= \frac{(191.5)^2+(248.4)^2-(118)^2}{2(191.5)(248.4)} \ \rightarrow \text{simplify}\\
\cos B &= \frac{84450.81}{95137.2} \rightarrow \text{divide}\\
\cos B &= 0.8877\\
\cos^{-1}(\cos B) &= \cos^{-1}(0.8877)\\
\angle B(\angle 2) &\approx 27.4^\circ
\end{aligned}$$

$$\begin{aligned}
\cos A &= \frac{b^2+d^2-a^2}{2bd}\\
\cos A &= \frac{b^2+d^2-a^2}{2bd} \ \rightarrow b = 118,\ d = 248.4,\ a = 191.5\\
\cos A &= \frac{(118)^2+(248.4)^2-(191.5)^2}{2(118)(248.4)} \ \rightarrow \text{simplify}\\
\cos A &= \frac{38954.31}{58622.4} \rightarrow \text{divide}\\
\cos A &= 0.6645\\
\cos^{-1}(\cos A) &= \cos^{-1}(0.6645)\\
\angle A(\angle 5) &\approx 48.4^\circ
\end{aligned}$$

$$\begin{aligned}
\angle D &= 180^\circ - (27.4^\circ + 48.4^\circ)\\
\angle D &= 180^\circ - (75.8^\circ)\\
\angle D(\angle 4) &= 104.2^\circ
\end{aligned}$$

Area of Triangle 2:

$$\begin{aligned}
K &= \frac{1}{2}ad\sin B\\
K &= \frac{1}{2}ad\sin B \rightarrow a = 191.5,\ d = 248.4,\ \angle E = 27.4^\circ\\
K &= \frac{1}{2}(191.5)(248.4)\sin(27.4^\circ) \rightarrow \text{simplify}\\
K &= \frac{1}{2}(191.5)(248.4)(0.4602)\\
K &= 10,945.5 \text{ square units}
\end{aligned}$$

Total Area:

18.062.8 square units $+12,133.0$ square units $+10,945.5$ square units $=41,141.3$ square units.

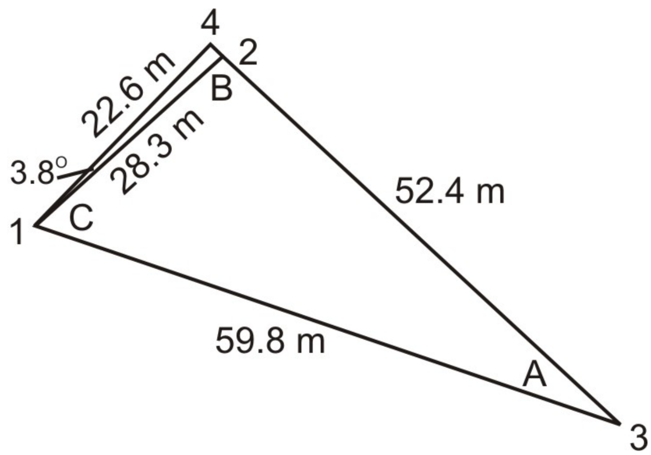
Measure of $\angle 2 = 49.2^\circ + 27.4^\circ + 46.8^\circ = 123.4^\circ$

Measure of $\angle 4 = 104.2^\circ + 60.2^\circ = 164.4^\circ$

Measure of $\angle 5 = 49.8^\circ + 48.4^\circ = 98.2^\circ$

7. This question cannot be answered. There is not enough information given.

8. If Island 4 is 22.6 miles from Island 1 and at a heading of 86.2° , then there is an angle of 3.8° made with Island 1 .



The distance from Island 2 to Island 4(D)

$$
\begin{aligned}
c^2 &= d^2 + b^2 - 2db\cos C \\
c^2 &= d^2 + b^2 - 2db\cos C \rightarrow d = 28.3,\ b = 22.6,\ \angle C = 3.8^\circ \\
c^2 &= (28.3)^2 + (22.6)^2 - 2(28.3)(22.6)\cos(3.8^\circ) \rightarrow \text{simplify} \\
c^2 &= 35.3023 \rightarrow \sqrt{\text{both sides}} \\
\sqrt{c^2} &= \sqrt{35.3023} \rightarrow \text{simplify} \\
c &\approx 5.9 \text{ units}
\end{aligned}
$$

The distance from Island 3 to Island 4 is 52.4 miles $+$ 5.9 miles $=$ 58.3 miles

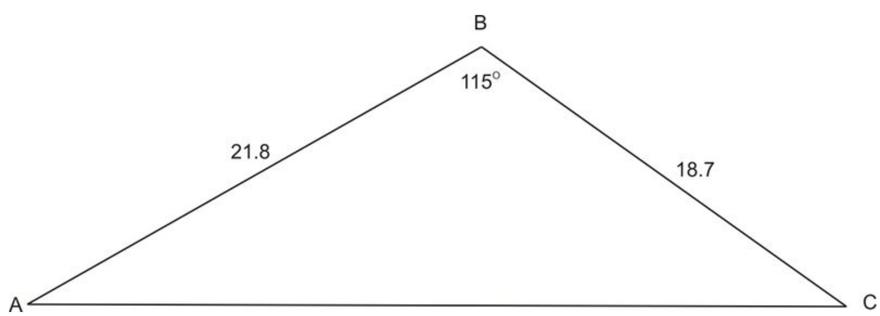
The angle formed by Island 3 with Islands 1 and 4

$$
\begin{aligned}
\cos B &= \frac{c^2 + d^2 - a^2}{2cd} \\
\cos B &= \frac{c^2 + d^2 - a^2}{2cd} \rightarrow a = 22.6,\ c = 58.3,\ d = 59.8 \\
\cos B &= \frac{(58.3)^2 + (59.8)^2 - (22.6)^2}{2(58.3)(59.8)} \rightarrow \text{simplify} \\
\cos B &= \frac{6464.17}{6972.68} \rightarrow \text{divide} \\
\cos B &= 0.9271 \\
\cos^{-1}(\cos B) &= \cos^{-1}(0.9271) \\
\angle B &\approx 22.0^\circ
\end{aligned}
$$

The angle formed by Island 4 with Islands 1 and 3

$$\begin{aligned}
\cos D &= \frac{a^2+c^2-d^2}{2ac}\\
\cos D &= \frac{a^2+c^2-d^2}{2ac} \rightarrow a = 22.6,\ c = 58.3,\ d = 59.8\\
\cos D &= \frac{(22.6)^2+(58.3)^2-(59.8)^2}{2(22.6)(58.3)} \rightarrow \text{simplify}\\
\cos D &= \frac{333.61}{2635.16} \rightarrow \text{divide}\\
\cos D &= 0.1266\\
\cos^{-1}(\cos D) &= \cos^{-1}(0.1266)\\
\angle D &\approx 82.7^\circ
\end{aligned}$$

9.a) The following diagram represents the problem. The Law of Cosines must be used to calculate the distance the ball must be shot to make it to the green in one shot.



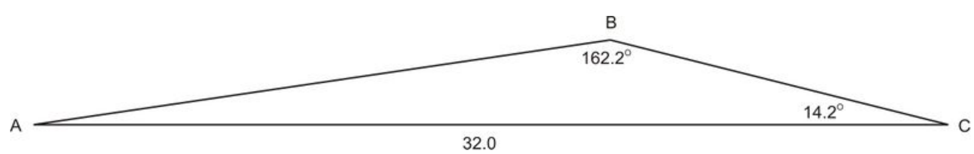
$$\begin{aligned}
b^2 &= a^2+c^2-2ac\cos B\\
b^2 &= a^2+c^2-2ac\cos B \rightarrow a = 187,\ c = 218,\ \angle B = 115^\circ\\
b^2 &= (187)^2+(218)^2-2(187)(218)\cos(115^\circ) \rightarrow \text{simplify}\\
b^2 &= 116949.9121 \rightarrow \sqrt{\text{both sides}}\\
\sqrt{b^2} &= \sqrt{116949.9121} \rightarrow \text{simplify}\\
b &\approx 342.0 \text{ yards}
\end{aligned}$$

b)

$$
\begin{aligned}
\frac{a}{\sin A} &= \frac{b}{\sin B}\\
\frac{a}{\sin A} &= \frac{b}{\sin B} \rightarrow a = 187,\ b = 342,\ \angle B = 115^\circ\\
\frac{187}{\sin A} &= \frac{342}{\sin(115^\circ)} \rightarrow \text{simplify}\\
187(\sin(115^\circ)) &= 342(\sin A) \rightarrow \text{simplify}\\
187(0.9063) &= 342(\sin A) \rightarrow \text{simplify}\\
169.4781 &= 342(\sin A) \rightarrow \text{solve}\\
\frac{169.4781}{342} &= \frac{\cancel{342}(\sin A)}{\cancel{342}} \rightarrow \text{solve}\\
0.4959 &= (\sin A)\\
\sin^{-1}(0.4956) &= \sin^{-1}(\sin A)\\
29.7^\circ &\approx \angle A
\end{aligned}
$$

He must hit the ball within an angle of 29.7° .

10.a) The following diagram represents the problem.



The degree of his slice is $180^\circ - (162.2^\circ + 14.2^\circ) = 3.6^\circ$

b)

$$
\begin{aligned}
\frac{b}{\sin B} &= \frac{c}{\sin C}\\
\frac{b}{\sin B} &= \frac{c}{\sin C} \rightarrow b = 320,\ \angle B = 162.2^\circ,\ \angle C = 14.2^\circ\\
\frac{320}{\sin(162.2^\circ)} &= \frac{c}{\sin(14.2^\circ)} \rightarrow \text{simplify}\\
320(\sin(14.2^\circ)) &= (\sin(162.2^\circ))c \rightarrow \text{simplify}\\
320(0.2453) &= (0.3057)c \rightarrow \text{simplify}\\
78.496 &= (0.3057)c \rightarrow \text{solve}\\
\frac{78.496}{0.3057} &= \frac{\cancel{(0.3057)}c}{\cancel{0.3057}} \rightarrow \text{solve}\\
256.8 \text{ yards} &\approx c
\end{aligned}
$$

c)

$$\begin{aligned}
\frac{a}{\sin A} &= \frac{c}{\sin C} \\
\frac{a}{\sin A} &= \frac{c}{\sin C} \rightarrow c = 256.8,\ \angle A = 3.6^\circ,\ \angle C = 14.2^\circ \\
\frac{a}{\sin(3.6^\circ)} &= \frac{256.8}{\sin(14.2^\circ)} \rightarrow \text{simplify} \\
(\sin(14.2^\circ))a &= 256.8(\sin(3.6^\circ)) \rightarrow \text{simplify} \\
(0.2453)a &= 256.8(0.0628) \rightarrow \text{simplify} \\
(0.2453)a &= 16.1270 \rightarrow \text{solve} \\
\frac{\cancel{(0.2453)}a}{\cancel{0.2453}} &= \frac{16.1270}{0.2453} = \rightarrow \text{solve} \\
a &\approx 65.7 \text{ yards}
\end{aligned}$$

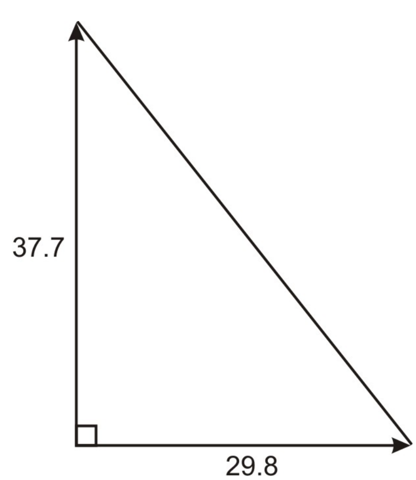
Vectors

Review Exercises:

1. Because $\vec{m}$ and $\vec{n}$ are perpendicular, the Pythagorean Theorem can be used determine the magnitude of the resultant vector. To determine the direction, the trigonometric ratios can be applied.

a)

$$\begin{aligned}
(h)^2 &= (s_1)^2 + (s_2)^2 \\
(h)^2 &= (29.8)^2 + (37.7)^2 \\
(h)^2 &= 2309.33 \\
\sqrt{h^2} &= \sqrt{2309.33} \\
h &\approx 48.1 \qquad \text{The magnitude is approximately 48.1 units.}
\end{aligned}$$



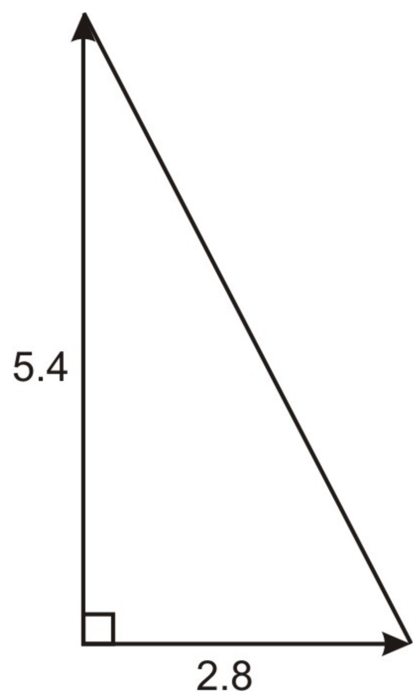
$$\begin{aligned}
\tan\theta &= \frac{\text{opp}}{\text{adj}} \\
\tan\theta &= \frac{37.7}{29.8} \\
\tan\theta &= 1.2651 \\
\tan^{-1}(\tan\theta) &= \tan^{-1}(1.2651) \\
\theta &\approx 51.7^\circ
\end{aligned}$$

The direction is approximately 51.7° .

b)

$$\begin{aligned}
(h)^2 &= (s_1)^2 + (s_2)^2 \\
(h)^2 &= (29.8)^2 + (5.4)^2 \\
(h)^2 &= 37 \\
\sqrt{h^2} &= \sqrt{37} \\
h &\approx 6.1
\end{aligned}$$

The magnitude is approximately 6.1 units.

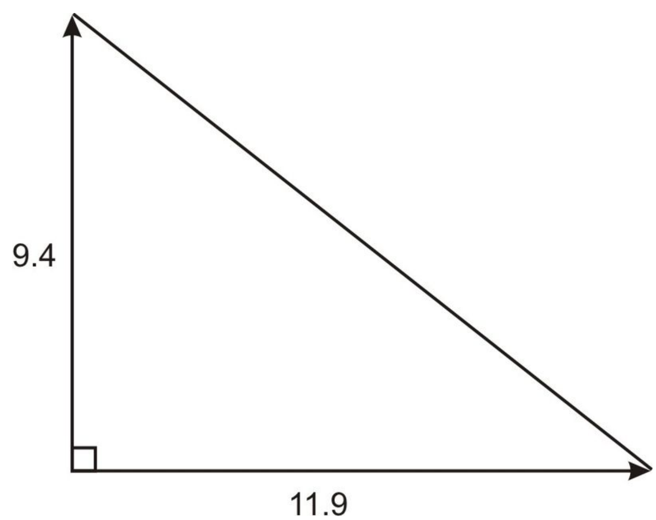


$$\begin{aligned}
\tan\theta &= \frac{\text{opp}}{\text{adj}} \\
\tan\theta &= \frac{5.4}{2.8} \\
\tan\theta &= 1.9286 \\
\tan^{-1}(\tan\theta) &= \tan^{-1}(1.9286) \\
\theta &\approx 62.6^\circ
\end{aligned}$$

The direction is approximately 62.6° .

c)

$$
\begin{aligned}
(h)^2 &= (s_1)^2 + (s_2)^2 \\
(h)^2 &= (11.9)^2 + (9.4)^2 \\
(h)^2 &= 229.97 \\
\sqrt{h^2} &= \sqrt{229.97} \\
h &\approx 15.2 \qquad \text{The magnitude is approximately 15.2 units.}
\end{aligned}
$$

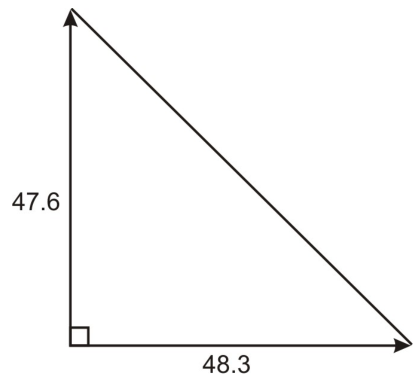


$$
\begin{aligned}
\tan\theta &= \frac{\text{opp}}{\text{adj}} \\
\tan\theta &= \frac{9.4}{11.9} \\
\tan\theta &= 0.7899 \\
\tan^{-1}(\tan\theta) &= \tan^{-1}(0.7899) \\
\theta &\approx 38.3^\circ
\end{aligned}
$$

The direction is approximately 38.3° .

d)

$$
\begin{aligned}
(h)^2 &= (s_1)^2 + (s_2)^2 \\
(h)^2 &= (48.3)^2 + (47.6)^2 \\
(h)^2 &= 4598.65 \\
\sqrt{(h)^2} &= \sqrt{4598.65} \\
h &\approx 67.8 \qquad \text{The magnitude is approximately 67.8 units.}
\end{aligned}
$$



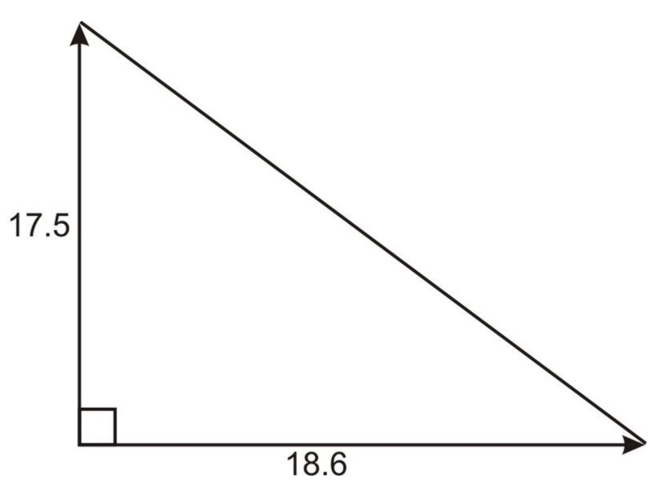
$$\tan\theta = \frac{\text{opp}}{\text{adj}}$$
$$\tan\theta = \frac{47.6}{48.3}$$
$$\tan\theta = 0.9855$$
$$\tan^{-1}(\tan\theta) = \tan^{-1}(0.9855)$$
$$\theta \approx 44.6^\circ$$

The magnitude is approximately 44.6°.

e)

$$(h)^2 = (s_1)^2 + (s_2)^2$$
$$(h)^2 = (18.6)^2 + (17.5)^2$$
$$(h)^2 = 652.21$$
$$\sqrt{(h)^2} = \sqrt{652.21}$$
$$h \approx 25.5$$

The magnitude is approximately 25.5 units.



$$\tan \theta = \frac{\text{opp}}{\text{adj}}$$
$$\tan \theta = \frac{17.5}{18.6}$$
$$\tan \theta = 0.9409$$
$$\tan^{-1}(\tan \theta) = \tan^{-1}(0.9409)$$
$$\theta \approx 43.3^\circ$$

The direction is approximately 43.3°

2.

TABLE 5.1:

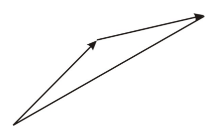
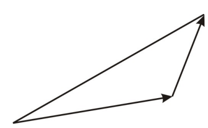
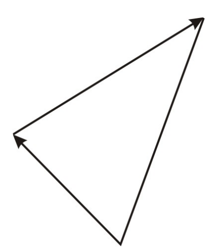
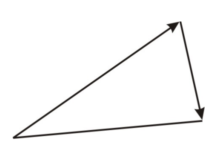
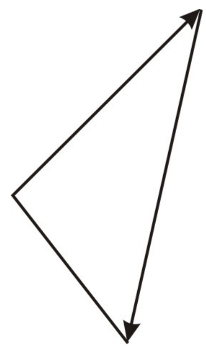
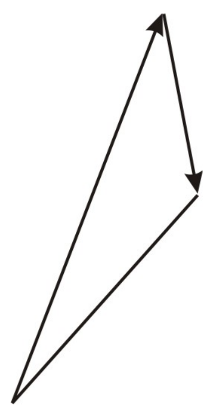
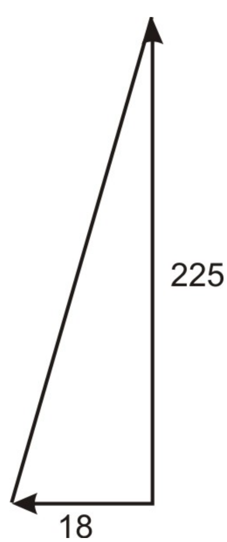
Operation	Diagram	Resultant
a) $\vec{a}+\vec{b}$		$\vec{a}+\vec{b} = 6\text{ cm}+3.2\text{ cm} = 9.2\text{ cm}$
b) $\vec{a}+\vec{d}$		$\vec{a}+\vec{d} = 6\text{ cm}+4.8\text{ cm} = 10.8\text{ cm}$
c) $\vec{c}+\vec{d}$		$\vec{c}+\vec{d} = 1.3\text{ cm}+4.8\text{ cm} = 6.1\text{ cm}$
d) $\vec{a}-\vec{d}$		$\vec{a}-\vec{d} = \vec{a}+(-\vec{d}) = 6\text{ cm} + (-4.8\text{ cm}) = 1.2\text{ cm}$

TABLE 5.1: (continued)

Operation	Diagram	Resultant
e) $\vec{b}-\vec{a}$		$\vec{b}-\vec{a}=\vec{b}+(-\vec{a})=3.2\text{ cm}-6\text{ cm}=2.8\text{ cm}$
f) $\vec{d}-\vec{c}$		$\vec{d}-\vec{c}=\vec{d}+(-\vec{c})=4.8\text{ cm}-1.3\text{ cm}=3.5\text{ cm}$

3. $|\vec{a}+\vec{b}|=|\vec{a}|+|\vec{b}|$ is true if and only if both vectors are positive.

4.

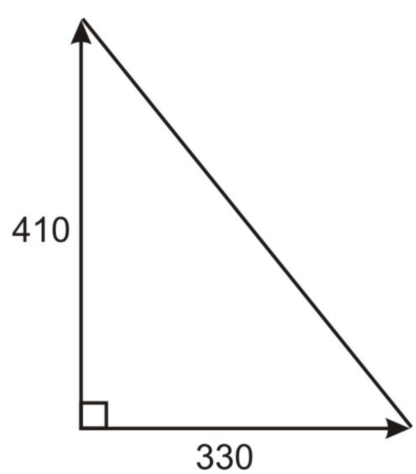


$$
\begin{aligned}
(h)^2 &= (s_1)^2 + (s_2)^2 \\
(h)^2 &= (225)^2 + (18)^2 \\
(h)^2 &= 50949 \\
\sqrt{(h)^2} &= \sqrt{50949} \\
h &\approx 225.7 \text{ mph} \qquad \text{The plane's speed is approximately 225.7 mph.}
\end{aligned}
$$

$$
\begin{aligned}
\tan\theta &= \frac{\text{opp}}{\text{adj}} \\
\tan\theta &= \frac{18}{225} \\
\tan\theta &= 0.08 \\
\tan^{-1}(\tan\theta) &= \tan^{-1}(0.08) \\
\theta &\approx 4.6^\circ \text{ NE}
\end{aligned}
$$

The direction is approximately 4.6° Northeast

5.



$$
\begin{aligned}
(h)^2 &= (s_1)^2 + (s_2)^2 \\
(h)^2 &= (330)^2 + (410)^2 \\
(h)^2 &= 277000 \\
\sqrt{(h)^2} &= \sqrt{277000} \\
h &\approx 526.3 \text{ Newtons}
\end{aligned}
$$

The magnitude is approximately 526.3 Newtons

$$\tan\theta = \frac{\text{opp}}{\text{adj}}$$
$$\tan\theta = \frac{410}{330}$$
$$\tan\theta = 1.2424$$
$$\tan^{-1}(\tan\theta) = \tan^{-1}(1.2424)$$
$$\theta \approx 51.2^\circ \text{ Northeast}$$

The direction is 51.2° *Northeast*.

6. To determine the magnitude and the direction of each vector in standard position, use the coordinates of the terminal point and the coordinates of the origin in the distance formula to calculate the magnitude. The $x-$ coordinate of the terminal point represents the horizontal distance and the $y-$ coordinate represents the vertical distance. These values can be used with the tangent function to determine the direction of the vector.

a)

$$|\vec{v}| = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$
$$|\vec{v}| = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2} \rightarrow (x_1,y_1) = (0,0)$$
$$\rightarrow (x_2,y_2) = (12,18)$$
$$|\vec{v}| = \sqrt{(12-0)^2+(18-0)^2} \rightarrow \text{simplify}$$
$$|\vec{v}| = \sqrt{(12)^2+(18)^2} \rightarrow \text{simplify}$$
$$|\vec{v}| = \sqrt{468} \rightarrow \text{simplify}$$
$$|\vec{v}| \approx 21.6$$

$$\tan\theta = \frac{\text{opp}}{\text{adj}}$$
$$\tan\theta = \frac{18}{12}$$
$$\tan\theta = 1.5$$
$$\tan^{-1}(\tan\theta) = \tan^{-1}(1.5)$$
$$\theta \approx 56.3^\circ$$

b)

$$|\vec{v}| = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$
$$|\vec{v}| = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2} \rightarrow (x_1,y_1) = (0,0)$$
$$\rightarrow (x_2,y_2) = (-3,6)$$
$$|\vec{v}| = \sqrt{(-3-0)^2+(6-0)^2} \rightarrow \text{simplify}$$
$$|\vec{v}| = \sqrt{(-3)^2+(6)^2} \rightarrow \text{simplify}$$
$$|\vec{v}| = \sqrt{45} \rightarrow \text{simplify}$$
$$|\vec{v}| \approx 67$$

$$\tan\theta = \frac{\text{opp}}{\text{adj}}$$
$$\tan\theta = \frac{6}{-3}$$
$$\tan\theta = -2.0$$
$$\tan^{-1}(\tan\theta) = \tan^{-1}(2.0)$$
$$\theta \approx 63.46^\circ \text{ but the tangent function is negative in the } 2^{\text{nd}} \text{ quadrant.}$$
$$\theta = 186^\circ - 63.4^\circ = 116.6^\circ$$

c)

$$|\vec{v}| = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$
$$|\vec{v}| = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2} \to (x_1,y_1) = (0,0)$$
$$\to (x_2,y_2) = (-1,-9)$$
$$|\vec{v}| = \sqrt{(-1-0)^2+(-9-0)^2} \to \text{simplify}$$
$$|\vec{v}| = \sqrt{(-1)^2+(-9)^2} \to \text{simplify}$$
$$|\vec{v}| = \sqrt{82} \to \text{simplify}$$
$$|\vec{v}| \approx 9.1$$

$$\tan\theta = \frac{\text{opp}}{\text{adj}}$$
$$\tan\theta = \frac{-9}{-1}$$
$$\tan\theta = 9.0$$
$$\tan^{-1}(\tan\theta) = \tan^{-1}(9.0)$$
$$\theta \approx 83.7^\circ$$

The angle is in the 3rd quadrant and has a value of $180^\circ + 83.7^\circ = 263.7^\circ$

d)

$$|\vec{v}| = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$
$$|\vec{v}| = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2} \to (x_1,y_1) = (0,0)$$
$$\to (x_2,y_2) = (3,-2)$$
$$|\vec{v}| = \sqrt{(3-0)^2+(-2-0)^2} \to \text{simplify}$$
$$|\vec{v}| = \sqrt{(3)^2+(-2)^2} \to \text{simplify}$$
$$|\vec{v}| = \sqrt{13} \to \text{simplify}$$
$$|\vec{v}| \approx 3.6$$

$$\tan\theta = \frac{\text{opp}}{\text{adj}}$$
$$\tan\theta = \frac{3}{-2}$$
$$\tan\theta = -1.5$$
$$\tan^{-1}(\tan\theta) = \tan^{-1}(1.5)$$
$$\theta \approx 56.3^\circ$$

The angle is in the 4^{th} quadrant and has a value of $270^\circ + 56.3^\circ = 326.3^\circ$

7. In order to determine the magnitude and direction of a vector that is not in standard position, the initial point must be translated to the origin and the terminal point translated the same number of units. For example a vector with an initial point $(2,4)$ and a terminal point $(8,6)$ will become $(2-2,4-4) = (0,0)$ and $(8-2,6-4) = (6,2)$. Once the points of the vector in standard position have been established, the distance formula and the tangent function can be applied to determine the magnitude and the direction.

a)

$$|\vec{v}| = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$
$$|\vec{v}| = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2} \rightarrow (x_1,y_1) = (0,0)$$
$$\rightarrow (x_2,y_2) = (6,2)$$
$$|\vec{v}| = \sqrt{(6-0)^2+(2-0)^2} \rightarrow \text{simplify}$$
$$|\vec{v}| = \sqrt{(6)^2+(2)^2} \rightarrow \text{simplify}$$
$$|\vec{v}| = \sqrt{40} \rightarrow \text{simplify}$$
$$|\vec{v}| \approx 6.3$$

$$\tan\theta = \frac{\text{opp}}{\text{adj}}$$
$$\tan\theta = \frac{2}{6}$$
$$\tan\theta = 0.3333$$
$$\tan^{-1}(\tan\theta) = \tan^{-1}(0.3333)$$
$$\theta \approx 18.4^\circ$$

b)

$$|\vec{v}| = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$
$$|\vec{v}| = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2} \rightarrow (x_1,y_1) = (0,0)$$
$$\rightarrow (x_2,y_2) = (-2,3)$$
$$|\vec{v}| = \sqrt{(-2-0)^2+(3-0)^2} \rightarrow \text{simplify}$$
$$|\vec{v}| = \sqrt{(-2)^2+(3)^2} \rightarrow \text{simplify}$$
$$|\vec{v}| = \sqrt{13} \rightarrow \text{simplify}$$
$$|\vec{v}| \approx 3.6$$

$$
\begin{aligned}
\tan\theta &= \frac{\text{opp}}{\text{adj}} \\
\tan\theta &= \frac{3}{-2} \\
\tan\theta &= -1.5 \\
\tan^{-1}(\tan\theta) &= \tan^{-1}(1.5) \\
\theta &\approx 56.3^\circ \text{ but the tangent function is negative in the } 2^{\text{nd}} \text{ quadrant} \\
\theta &= 180^\circ - 56.3^\circ = 123.7^\circ
\end{aligned}
$$

c)

$$
\begin{aligned}
|\vec{v}| &= \sqrt{(x_2-x_1)^2+(y_2-y_1)^2} \\
|\vec{v}| &= \sqrt{(x_2-x_1)^2+(y_2-y_1)^2} \rightarrow (x_1,y_1) = (0,0) \\
&\qquad\qquad\qquad\qquad\qquad \rightarrow (x_2,y_2) = (16,-18) \\
|\vec{v}| &= \sqrt{(16-0)^2+(-18-0)^2} \rightarrow \text{simplify} \\
|\vec{v}| &= \sqrt{(16)^2+(-18)^2} \rightarrow \text{simplify} \\
|\vec{v}| &= \sqrt{580} \rightarrow \text{simplify} \\
|\vec{v}| &\approx 24.1
\end{aligned}
$$

$$
\begin{aligned}
\tan\theta &= \frac{\text{opp}}{\text{adj}} \\
\tan\theta &= \frac{-18}{16} \\
\tan\theta &= -1.125 \\
\tan^{-1}(\tan\theta) &= \tan^{-1}(1.125) \\
\theta &\approx 48.4^\circ
\end{aligned}
$$

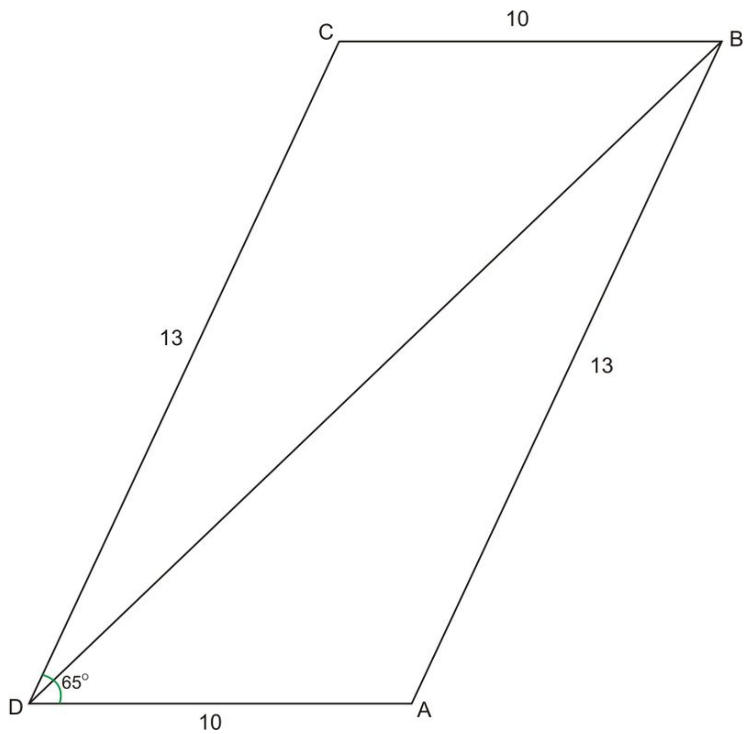
The angle is in the 4^{th} quadrant and has a value of $360^\circ - 48.4^\circ = 311.6^\circ$

d)

$$
\begin{aligned}
|\vec{v}| &= \sqrt{(x_2-x_1)^2+(y_2-y_1)^2} \\
|\vec{v}| &= \sqrt{(x_2-x_1)^2+(y_2-y_1)^2} \rightarrow (x_1,y_1) = (0,0) \\
&\qquad\qquad\qquad\qquad\qquad \rightarrow (x_2,y_2) = (10,10) \\
|\vec{v}| &= \sqrt{(10-0)^2+(10-0)^2} \rightarrow \text{simplify} \\
|\vec{v}| &= \sqrt{(10)^2+(10)^2} \rightarrow \text{simplify} \\
|\vec{v}| &= \sqrt{200} \rightarrow \text{simplify} \\
|\vec{v}| &\approx 14.1
\end{aligned}
$$

$$\tan\theta = \frac{\text{opp}}{\text{adj}}$$
$$\tan\theta = \frac{10}{10}$$
$$\tan\theta = 1.0$$
$$\tan^{-1}(\tan\theta) = \tan^{-1}(1.0)$$
$$\theta \approx 45^\circ$$

8. To determine the magnitude of the resultant vector and the angle it makes with a, the parallelogram method will have to be used. The opposite angles of a parallelogram are congruent as are the opposite sides. The Law of Cosines can be used to calculate the magnitude of the resultant vector.



a) If $\angle CDA = 65^\circ$ then $\angle ABC = 65^\circ$

$\angle BCD = \angle BAD$

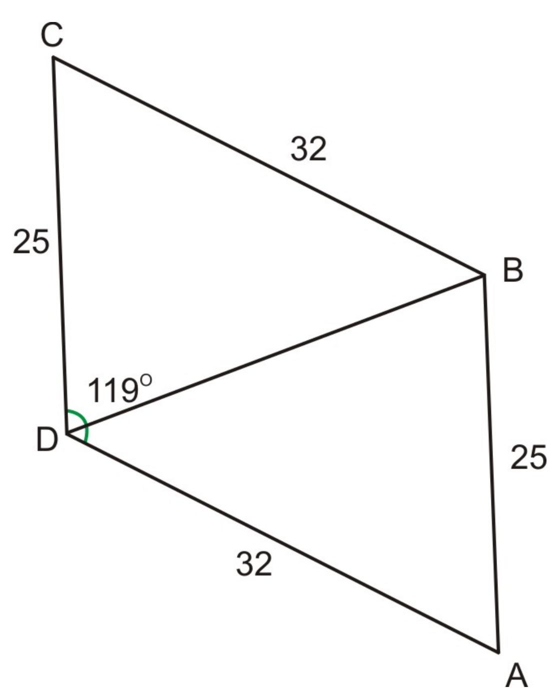
$$\angle CDA + \angle ABC + \angle BCD + \angle BAD = 360^\circ$$
$$2\angle CDA + 2\angle BCD = 360^\circ$$
$$2(65^\circ) + 2\angle BCD = 360^\circ$$
$$130^\circ + 2\angle BCD = 360^\circ$$
$$2\angle BCD = 360^\circ - 130^\circ$$
$$2\angle BCD = 230^\circ$$
$$\frac{\cancel{2}\angle BCD}{\cancel{2}} = \frac{230^\circ}{2}$$
$$\angle BCD = 115^\circ$$

In $\triangle BAD$

$$
\begin{aligned}
a^2 &= b^2+d^2-2bd\cos A\\
a^2 &= b^2+d^2-2bd\cos A \rightarrow b=10,\ d=13,\ \angle A=115^\circ\\
a^2 &= (10)^2+(13)^2-2(10)(13)\cos(115^\circ) \rightarrow \text{simplify}\\
a^2 &= 378.8807 \rightarrow \sqrt{\text{both sides}}\\
\sqrt{a^2} &= \sqrt{378.8807} \rightarrow \text{simplify}\\
a &\approx 19.5 \text{ units}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\sin A}{a} &= \frac{\sin D}{d}\\
\frac{\sin A}{a} &= \frac{\sin D}{d} \rightarrow \angle A=115^\circ,\ a=19.5,\ \angle d=13\\
\frac{\sin(115^\circ)}{19.5} &= \frac{\sin D}{13} \rightarrow \text{simplify}\\
\sin(115^\circ)(13) &= (19.5)\sin D \rightarrow \text{simplify}\\
(0.9063)(13) &= (19.5)(\sin D) \rightarrow \text{simplify}\\
11.7820 &= (19.5)\sin D \rightarrow \text{solve}\\
\frac{11.7820}{19.5} &= \frac{\cancel{(19.5)}\sin D}{\cancel{19.5}} \rightarrow \text{solve}\\
0.6042 &= \sin D \rightarrow \text{solve}\\
\sin^{-1}(0.6042) &= \sin^{-1}(\sin D)\\
37.2^\circ &\approx \angle D
\end{aligned}
$$

b)



If $\angle CDA = 119^\circ$ then $\angle ABC = 119^\circ$

$\angle BCD = \angle BAD$

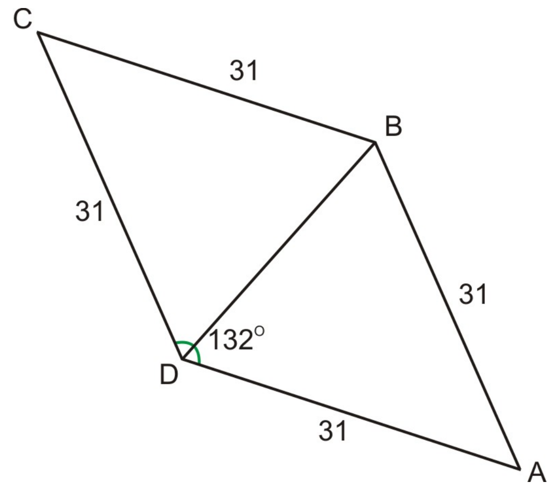
$$\begin{aligned}
\angle CDA + \angle ABC + \angle BCD + \angle BAD &= 360^\circ \\
2\angle CDA + 2\angle BCD &= 360^\circ \\
2(119^\circ) + 2\angle BCD &= 360^\circ \\
238^\circ + 2\angle BCD &= 360^\circ \\
2\angle BCD &= 360^\circ - 238^\circ \\
2\angle BCD &= 122^\circ \\
\frac{\cancel{2}\angle BCD}{\cancel{2}} &= \frac{122^\circ}{2} \\
\angle BCD &= 61^\circ
\end{aligned}$$

In $\triangle BAD$

$$\begin{aligned}
a^2 &= b^2 + d^2 - 2bd\cos A \\
a^2 &= b^2 + d^2 - 2bd\cos A \rightarrow b = 32,\ d = 25,\ \angle A = 61^\circ \\
a^2 &= (32)^2 + (25)^2 - 2(32)(25)\cos(61^\circ) \rightarrow \text{simplify} \\
a^2 &= 873.3046 \rightarrow \sqrt{\text{both sides}} \\
\sqrt{a^2} &= \sqrt{873.3046} \rightarrow \text{simplify} \\
a &\approx 29.6 \text{ units}
\end{aligned}$$

$$\begin{aligned}
\frac{\sin A}{a} &= \frac{\sin D}{d} \\
\frac{\sin A}{a} &= \frac{\sin D}{d} \rightarrow \angle A = 61^\circ,\ a = 29.6,\ \angle d = 32 \\
\frac{\sin(61^\circ)}{29.6} &= \frac{\sin D}{32} \rightarrow \text{simplify} \\
\sin(61^\circ)(32) &= (29.6)\sin D \rightarrow \text{simplify} \\
(0.8746)(32) &= (29.6)(\sin D) \rightarrow \text{simplify} \\
27.9872 &= (29.6)\sin D \rightarrow \text{solve} \\
\frac{27.9872}{29.6} &= \frac{\cancel{(29.6)}\sin D}{\cancel{29.6}} \rightarrow \text{solve} \\
0.9455 &= \sin D \rightarrow \text{solve} \\
\sin^{-1}(0.9455) &= \sin^{-1}(\sin D) \\
71.0^\circ &\approx \angle D
\end{aligned}$$

c)



If $\angle CDA = 132^\circ$ then $\angle ABC = 132^\circ$

$\angle BCD = \angle BAD$

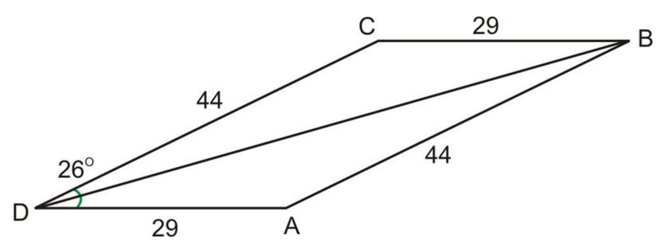
$$\begin{aligned}
\angle CDA + \angle ABC + \angle BCD + \angle BAD &= 360^\circ \\
2\angle CDA + 2\angle BCD &= 360^\circ \\
2(132^\circ) + 2\angle BCD &= 360^\circ \\
264^\circ + 2\angle BCD &= 360^\circ \\
2\angle BCD &= 360^\circ - 264^\circ \\
2\angle BCD &= 96^\circ \\
\frac{\cancel{2}\angle BCD}{\cancel{2}} &= \frac{96^\circ}{2} \\
\angle BCD &= 48^\circ
\end{aligned}$$

In $\triangle BAD$

$$\begin{aligned}
a^2 &= b^2 + d^2 - 2bd\cos A \\
a^2 &= b^2 + d^2 - 2bd\cos A \rightarrow b = 31,\ d = 31,\ \angle A = 48^\circ \\
a^2 &= (31)^2 + (31)^2 - 2(31)(31)\cos(48^\circ) \rightarrow \text{simplify} \\
a^2 &= 635.9310 \rightarrow \sqrt{\text{both sides}} \\
\sqrt{a^2} &= \sqrt{635.9310} \rightarrow \text{simplify} \\
a &\approx 25.2 \text{ units}
\end{aligned}$$

$$\begin{aligned}
\frac{\sin A}{a} &= \frac{\sin D}{d} \\
\frac{\sin A}{a} &= \frac{\sin D}{d} \rightarrow \angle A = 48^\circ,\ a = 25.2,\ d = 31 \\
\frac{\sin(48^\circ)}{25.2} &= \frac{\sin D}{31} \rightarrow \text{simplify} \\
\sin(48^\circ)(31) &= (25.5)\sin D \rightarrow \text{simplify} \\
(0.7431)(31) &= (25.2)(\sin D) \rightarrow \text{simplify} \\
23.0361 &= (25.2)\sin D \rightarrow \text{solve} \\
\frac{23.0361}{25.2} &= \frac{\cancel{(25.2)}\sin D}{\cancel{25.2}} \rightarrow \text{solve} \\
0.9141 &= \sin D \rightarrow \text{solve} \\
\sin^{-1}(0.9141) &= \sin^{-1}(\sin D) \\
66.1^\circ &\approx \angle D
\end{aligned}$$

d)



If $\angle CDA = 26^\circ$ then $\angle ABC = 26^\circ$

$\angle BCD = \angle BAD$

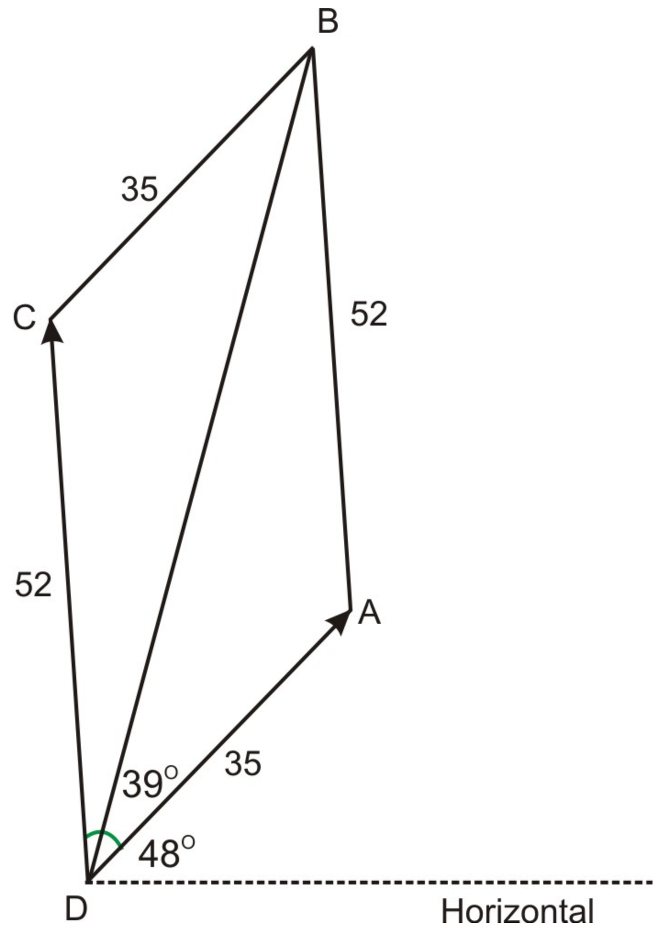
$$\begin{aligned}
\angle CDA + \angle ABC + \angle BCD + \angle BAD &= 360^\circ \\
2\angle CDA + 2\angle BCD &= 360^\circ \\
2(26^\circ) + 2\angle BCD &= 360^\circ \\
52^\circ + 2\angle BCD &= 360^\circ \\
2\angle BCD &= 360^\circ - 52^\circ \\
2\angle BCD &= 308^\circ \\
\frac{\cancel{2}\angle BCD}{\cancel{2}} &= \frac{308^\circ}{2} \\
\angle BCD &= 154^\circ
\end{aligned}$$

In $\triangle BAD$

$$
\begin{aligned}
a^2 &= b^2 + d^2 - 2bd\cos A \\
a^2 &= b^2 + d^2 - 2bd\cos A \rightarrow b = 29,\ d = 44,\ \angle A = 154^\circ \\
a^2 &= (29)^2 + (44)^2 - 2(29)(44)\cos(15.4^\circ) \rightarrow \text{simplify} \\
a^2 &= 5070.7224 \rightarrow \sqrt{\text{both sides}} \\
\sqrt{a^2} &= \sqrt{5070.7224} \rightarrow \text{simplify} \\
a &\approx 71.2 \text{ units}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\sin A}{a} &= \frac{\sin D}{d} \\
\frac{\sin A}{a} &= \frac{\sin D}{d} \rightarrow \angle A = 154^\circ,\ a = 71.2,\ d = 29 \\
\frac{\sin(154^\circ)}{71.2} &= \frac{\sin D}{29} \rightarrow \text{simplify} \\
\sin(154^\circ)(29) &= (71.2)\sin D \rightarrow \text{simplify} \\
(0.4384)(29) &= (71.2)(\sin D) \rightarrow \text{simplify} \\
12.7136 &= (71.2)\sin D \rightarrow \text{solve} \\
\frac{12.7136}{71.2} &= \frac{\cancel{(71.2)}\sin D}{\cancel{71.2}} \rightarrow \text{solve} \\
0.1786 &= \sin D \rightarrow \text{solve} \\
\sin^{-1}(0.1786) &= \sin^{-1}(\sin D) \\
10.3^\circ &\approx \angle D
\end{aligned}
$$

9. To solve this problem, it must be noted that the angle of 48° is made with the horizontal and is located outside of the parallelogram. The angle inside of the parallelogram is the difference between the angle made with the horizontal by car A and the angle made with the horizontal by car B. This angle is 39° . The solution may now be completed by using the Law of Cosines to determine the magnitude and the Law of Sines to calculate the direction of the resultant.



If $\angle CDA = 39^\circ$ then $\angle ABC = 39^\circ$

$\angle BCD = \angle BAD$

$$\begin{aligned}
\angle CDA + \angle ABC + \angle BCD + \angle BAD &= 360^\circ \\
2\angle CDA + 2\angle BCD &= 360^\circ \\
2(39^\circ) + 2\angle BCD &= 360^\circ \\
78^\circ + 2\angle BCD &= 360^\circ \\
2\angle BCD &= 360^\circ - 78^\circ \\
2\angle BCD &= 282^\circ \\
\frac{\cancel{2}\angle BCD}{\cancel{2}} &= \frac{282^\circ}{2} \\
\angle BCD &= 141^\circ
\end{aligned}$$

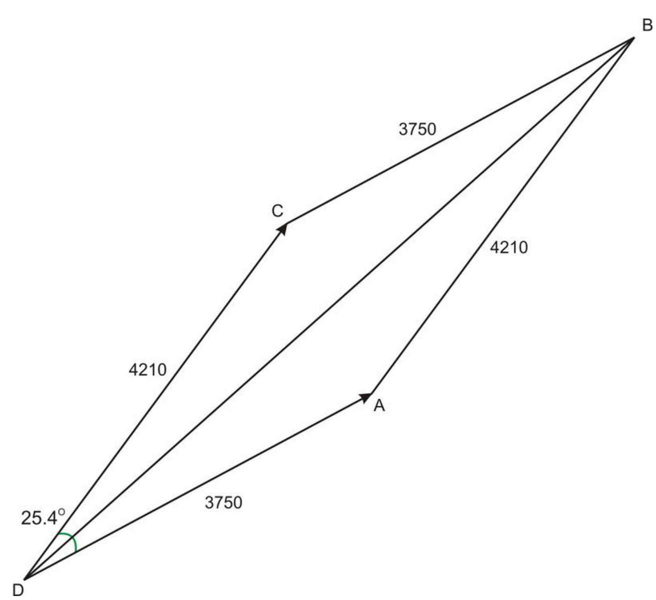
In $\triangle BAD$

$$
\begin{aligned}
a^2 &= b^2+d^2-2bd\cos A\\
a^2 &= b^2+d^2-2bd\cos A \rightarrow b=35,\ d=52,\ \angle A=141^\circ\\
a^2 &= (35)^2+(52)^2-2(35)(52)\cos(141^\circ) \rightarrow \text{simplify}\\
a^2 &= 6757.8113 \rightarrow \sqrt{\text{both sides}}\\
\sqrt{a^2} &= \sqrt{6757.8113} \rightarrow \text{simplify}\\
a &\approx 82.2 \text{ units}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\sin A}{a} &= \frac{\sin D}{d}\\
\frac{\sin A}{a} &= \frac{\sin D}{d} \rightarrow \angle A=141^\circ,\ a=82.2,\ d=52\\
\frac{\sin(141^\circ)}{82.2} &= \frac{\sin D}{52} \rightarrow \text{simplify}\\
\sin(141^\circ)(52) &= (82.2)\sin D \rightarrow \text{simplify}\\
(0.6293)(52) &= (82.2)(\sin D) \rightarrow \text{simplify}\\
32.7236 &= (82.2)\sin D \rightarrow \text{solve}\\
\frac{32.7236}{82.2} &= \frac{\cancel{(82.2)}\sin D}{\cancel{82.2}} \rightarrow \text{solve}\\
0.3981 &= \sin D \rightarrow \text{solve}\\
\sin^{-1}(0.3981) &= \sin^{-1}(\sin D)\\
23.5^\circ &\approx \angle D
\end{aligned}
$$

The direction is this result plus the angle that car A makes with the horizontal. $23.5^\circ + 48^\circ = 71.5^\circ$

10. To solve this problem, the Law of Cosines must be used to determine the magnitude of the resultant and the Law of Sines to calculate the direction that the resultant makes with the smaller force.



If $\angle CDA = 25.4^\circ$ then $\angle ABC = 25.4^\circ$

$\angle BCD = \angle BAD$

$$\begin{aligned}
\angle CDA + \angle ABC + \angle BCD + \angle BAD &= 360^\circ \\
2\angle CDA + 2\angle BCD &= 360^\circ \\
2(25.4^\circ) + 2\angle BCD &= 360^\circ \\
50.8^\circ + 2\angle BCD &= 360^\circ \\
2\angle BCD &= 360^\circ - 50.8^\circ \\
2\angle BCD &= 309.2^\circ \\
\frac{\cancel{2}\angle BCD}{\cancel{2}} &= \frac{309.2^\circ}{2} \\
\angle BCD &= 154.6^\circ
\end{aligned}$$

In $\triangle BAD$

$$\begin{aligned}
a^2 &= b^2 + d^2 - 2bd\cos A \\
a^2 &= b^2 + d^2 - 2bd\cos A \rightarrow b = 3750,\ d = 4210,\ \angle A = 154.6^\circ \\
a^2 &= (3750)^2 + (4210)^2 - 2(3750)(4210)\cos(154.6^\circ) \rightarrow \text{simplify} \\
a^2 &= 60309411.87 \rightarrow \sqrt{\text{both sides}} \\
\sqrt{a^2} &= \sqrt{60309411.87} \rightarrow \text{simplify} \\
a &\approx 7,765.9 \approx 7,766 \text{ lbs.}
\end{aligned}$$

$$\begin{aligned}
\frac{\sin A}{a} &= \frac{\sin D}{d} \\
\frac{\sin A}{a} &= \frac{\sin D}{d} \rightarrow \angle A = 154.6^\circ,\ a = 7766, d = 3750 \\
\frac{\sin(154.6^\circ)}{7766} &= \frac{\sin D}{3750} \rightarrow \text{simplify} \\
\sin(154.6^\circ)(3750) &= (7766)\sin D \rightarrow \text{simplify} \\
(0.4289)(3750) &= (7766)(\sin D) \rightarrow \text{simplify} \\
1608.375 &= (7766)\sin D \rightarrow \text{solve} \\
\frac{1608.375}{7766} &= \frac{\cancel{(7766)}\sin D}{\cancel{7766}} \rightarrow \text{solve} \\
0.2071 &= \sin D \rightarrow \text{solve} \\
\sin^{-1}(0.2071) &= \sin^{-1}(\sin D) \\
12^\circ &\approx \angle D
\end{aligned}$$

Component Vectors

Review Exercises:

1. To determine the resulting ordered pair, simply apply scalar multiplication.

a)

$$\begin{aligned}
&\vec{a} = 2\vec{b}\\
&\vec{a} = 2\vec{b} \rightarrow \vec{b} = (0,0) \text{ to } (5,4)\\
&\vec{a} = 2\vec{b} \rightarrow 2(5,4) = (10,8)\\
&\vec{a} = (10,8)\\
&\vec{a} = (0,0) \text{ to } (10,8)
\end{aligned}$$

b)

$$\begin{aligned}
&\vec{a} = -\frac{1}{2}\vec{c}\\
&\vec{a} = -\frac{1}{2}\vec{c} \rightarrow c = (0,0) \text{ to } (-3,7)\\
&\vec{a} = -\frac{1}{2}\vec{c} \rightarrow c = -\frac{1}{2}(-3,7) = (1.5,-3.5)\\
&\vec{a} = (1.5,-3.5)\\
&\vec{a} = (0,0) \text{ to } (1.5,-3.5)
\end{aligned}$$

c)

$$\begin{aligned}
&\vec{a} = 0.6\vec{b}\\
&\vec{a} = 0.6\vec{b} \rightarrow \vec{b} = (0,0) \text{ to } (5,4)\\
&\vec{a} = 0.6\vec{b} \rightarrow \vec{b} = 0.6(5,4) = (3,2.4)\\
&\vec{a} = (3,2.4)\\
&\vec{a} = (0,0) \text{ to } (3,2.4)
\end{aligned}$$

d)

$$\begin{aligned}
&\vec{a} = -3\vec{b}\\
&\vec{a} = -3\vec{b} \rightarrow \vec{b} = (0,0) \text{ to } (5,4)\\
&\vec{a} = -3\vec{b} \rightarrow b = -3(5,4) = (-15,-12)\\
&\vec{a} = (-15,-12)\\
&\vec{a} = (0,0) \text{ to } (-15,-12)
\end{aligned}$$

2. To determine the magnitude of the vertical and horizontal components of these vectors, add the absolute values of the coordinates necessary to return the initial point to the origin with the absolute value of the coordinates of the terminal point.

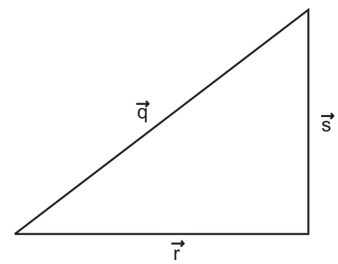
a) horizontal $= |3|+|2| = 5$ vertical $= |-8|+|-1| = 9$

b) horizontal $= |-7|+|11| = 18$ vertical $= |-13|+|19| = 32$

c) horizontal $= |-4.2|+|-1.3| = 5.5$ vertical $= |6.8|+|-9.4| = 16.2$

d) horizontal $= |-5.23|+|-0.237| = 5.467$ vertical $= |-4.98|+|0| = 4.98$

3. To determine the magnitude of the horizontal and vertical components if the resultant vector's magnitude and direction are given, use the trigonometric ratio for cosine to determine the magnitude of the horizontal component and the trigonometric ratio for sine to determine the magnitude of the vertical component. When calculating these values, consider the direction to be an angle in standard position and the magnitude of the resultant to be the hypotenuse $\vec{q}$ of the right triangle $\vec{q}\vec{r}\vec{s}$.



a)

$$\begin{aligned}\cos 35^\circ &= \frac{|\vec{r}|}{|\vec{q}|} = \frac{r}{q}\\ \cos 35^\circ &= \frac{r}{75}\\ 0.8192 &= \frac{r}{75}\\ 75(0.8192) &= \cancel{75}\left(\frac{r}{\cancel{75}}\right)\\ |61.4| &\approx r(\text{horizontal})\\ 61.4 &\approx r(\text{horizontal})\end{aligned}$$

$$\begin{aligned}\sin 35^\circ &= \frac{|\vec{s}|}{|\vec{q}|} = \frac{s}{q}\\ \sin 35^\circ &= \frac{s}{75}\\ 0.5736 &= \frac{s}{75}\\ 75(0.5736) &= \cancel{75}\left(\frac{s}{\cancel{75}}\right)\\ |43| &\approx s(\text{vertical})\\ 43 &\approx s(\text{vertical})\end{aligned}$$

b)

$$\begin{aligned}\cos 162^\circ &= \frac{|\vec{r}|}{|\vec{q}|} = \frac{r}{q}\\ \cos 162^\circ &= \frac{r}{3.4}\\ -0.9511 &= \frac{r}{3.4}\\ 3.4(-0.9511) &= \cancel{3.4}\left(\frac{r}{\cancel{3.4}}\right)\\ |-3.2| &\approx r(\text{horizontal})\\ 3.2 &\approx r(\text{horizontal})\end{aligned}$$

$$\begin{aligned}\sin 162^\circ &= \frac{|\vec{s}|}{|\vec{q}|} = \frac{s}{q}\\ \sin 162^\circ &= \frac{s}{3.4}\\ 0.3090 &= \frac{s}{3.4}\\ 3.4(0.3090) &= \cancel{3.4}\left(\frac{s}{\cancel{3.4}}\right)\\ |1.1| &\approx s(\text{vertical})\\ 1.1 &\approx s(\text{vertical})\end{aligned}$$

c)

$$\begin{aligned}\cos 12^\circ &= \frac{|\vec{r}|}{|\vec{q}|} = \frac{r}{q}\\ \cos 12^\circ &= \frac{r}{15.9}\\ 0.9781 &= \frac{r}{15.9}\\ 15.9(0.9781) &= \cancel{15.9}\left(\frac{r}{\cancel{15.9}}\right)\\ |15.6| &\approx r(\text{horizontal})\\ 15.6 &\approx r(\text{horizontal})\end{aligned}$$

$$\begin{aligned}\sin 12^\circ &= \frac{|\vec{s}|}{|\vec{q}|} = \frac{s}{q}\\ \sin 12^\circ &= \frac{s}{15.9}\\ 0.2079 &= \frac{s}{15.9}\\ 15.9(0.2079) &= \cancel{15.9}\left(\frac{s}{\cancel{15.9}}\right)\\ |3.3| &\approx s(\text{vertical})\\ 3.3 &\approx s(\text{vertical})\end{aligned}$$

d)

$$\cos 223^\circ = \frac{|\vec{r}|}{|\vec{q}|} = \frac{r}{q}$$
$$\cos 223^\circ = \frac{r}{189.27}$$
$$-0.7314 = \frac{r}{189.27}$$
$$189.27(-0.7314) = \cancel{189.27}\left(\frac{r}{\cancel{189.27}}\right)$$
$$|-138.4| \approx r(\text{horizontal})$$
$$138.4 \approx r(\text{horizontal})$$

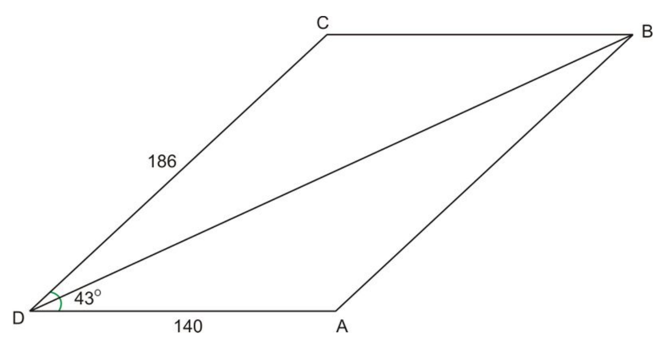
$$\sin 223^\circ = \frac{|\vec{s}|}{|\vec{q}|} = \frac{s}{q}$$
$$\sin 223^\circ = \frac{s}{189.27}$$
$$-0.6820 = \frac{s}{189.27}$$
$$189.27(-0.6820) = \cancel{189.27}\left(\frac{s}{\cancel{189.27}}\right)$$
$$|-129.1| \approx s(\text{vertical})$$
$$129.1 \approx s(\text{vertical})$$

4. To determine the magnitude and the direction of the resultant vector, the Pythagorean Theorem can be use to calculate the magnitude and the trigonometric ratio for sine can be used to determine the angle that it makes with the smaller force.

$$(\vec{q})^2 = (\vec{r})^2 + (\vec{s})^2$$
$$(\vec{q})^2 = (\overrightarrow{32.1})^2 + (\overrightarrow{8.50})^2$$
$$(\vec{q})^2 = 1102.66$$
$$\sqrt{(\vec{q})^2} = \sqrt{1102.66}$$
$$\vec{q} \approx 33.2 \text{ Newtons}$$

$$\sin x = \frac{|\vec{s}|}{|\vec{q}|} = \frac{s}{q}$$
$$\sin x = \frac{8.50}{33.2}$$
$$\sin x = 0.2560$$
$$\sin^{-1}(\sin x) = \sin^{-1}(0.2560)$$
$$x \approx 14.8^\circ$$

5. To determine the magnitude of the resultant and the angle it makes with the larger force, the parallelogram method must be used. Once the diagram has been sketched, the Law of Cosines and the Law of Sines can be used.



If $\angle CDA = 43^\circ$ then $\angle ABC = 43^\circ$

$$\angle BCD = \angle BAD$$

$$\begin{aligned}
\angle CDA + \angle ABC + \angle BCD + \angle BAD &= 360^\circ \\
2\angle CDA + 2\angle BCD &= 360^\circ \\
2(43^\circ) + 2\angle BCD &= 360^\circ \\
86^\circ + 2\angle BCD &= 360^\circ \\
2\angle BCD &= 360^\circ - 86^\circ \\
2\angle BCD &= 274^\circ \\
\frac{\cancel{2}\angle BCD}{\cancel{2}} &= \frac{274^\circ}{2} \\
\angle BCD &= 137^\circ
\end{aligned}$$

In $\triangle BAD$

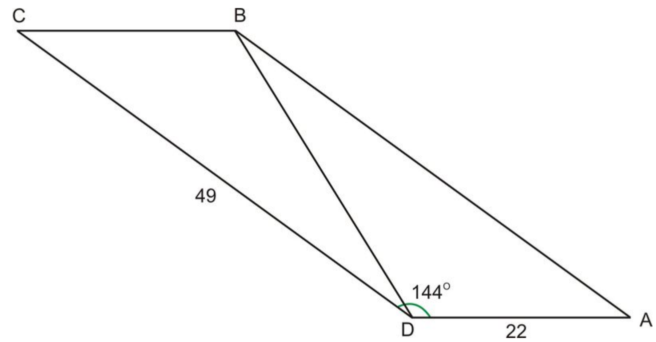
$$\begin{aligned}
a^2 &= b^2 + d^2 - 2bd\cos A \\
a^2 &= b^2 + d^2 - 2bd\cos A \to b = 140,\ d = 186,\ \angle A = 137^\circ \\
a^2 &= (140)^2 + (186)^2 - 2(140)(186)\cos(137^\circ) \to \text{simplify} \\
a^2 &= 92284.9008 \to \sqrt{\text{both sides}} \\
\sqrt{a^2} &= \sqrt{92284.9008} \to \text{simplify} \\
a &\approx 303.8 \approx 304 \text{ Newtons}
\end{aligned}$$

$$\begin{aligned}
\frac{\sin A}{a} &= \frac{\sin D}{d} \\
\frac{\sin A}{a} &= \frac{\sin D}{d} \to \angle A = 137^\circ,\ a = 304,\ d = 186 \\
\frac{\sin(137^\circ)}{304} &= \frac{\sin D}{186} \to \text{simplify} \\
\sin(137^\circ)(186) &= (304)\sin D \to \text{simplify} \\
(0.6820)(186) &= (304)(\sin D) \to \text{simplify} \\
126.852 &= (304)\sin D \to \text{solve} \\
\frac{126.852}{304} &= \frac{\cancel{(304)}\sin D}{\cancel{304}} \to \text{solve} \\
0.4173 &= \sin D \to \text{solve} \\
\sin^{-1}(0.4173) &= \sin^{-1}(\sin D) \\
24.7^\circ &\approx \angle D
\end{aligned}$$

This angle is counterclockwise from the smaller force.

6. To determine the magnitude of the resultant and the angle it makes with $\vec{a}$, the parallelogram method must be used. Once the diagram has been sketched, the Law of Cosines and the Law of Sines can be used.

a)



If $\angle CDA = 144^\circ$ then $\angle ABC = 144^\circ$

$\angle BCD = \angle BAD$

$$\begin{aligned}
\angle CDA + \angle ABC + \angle BCD + \angle BAD &= 360^\circ \\
2\angle CDA + 2\angle BCD &= 360^\circ \\
2(144^\circ) + 2\angle BCD &= 360^\circ \\
288^\circ + 2\angle BCD &= 360^\circ \\
2\angle BCD &= 360^\circ - 288^\circ \\
2\angle BCD &= 72^\circ \\
\frac{\cancel{2}\angle BCD}{\cancel{2}} &= \frac{72^\circ}{2} \\
\angle BCD &= 36^\circ
\end{aligned}$$

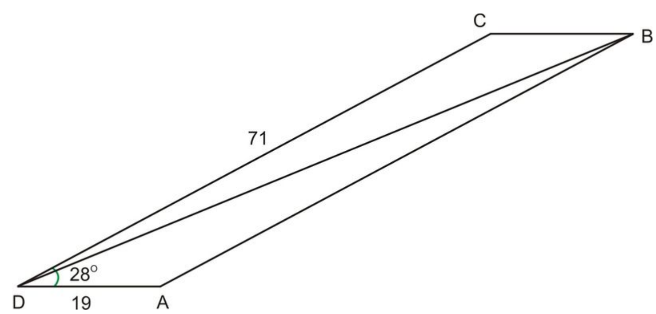
In $\triangle BAD$

$$\begin{aligned}
a^2 &= b^2 + d^2 - 2bd\cos A \\
a^2 &= b^2 + d^2 - 2bd\cos A \rightarrow b = 22,\ d = 49,\ \angle A = 36^\circ \\
a^2 &= (22)^2 + (49)^2 - 2(22)(49)\cos(36^\circ) \rightarrow \text{simplify} \\
a^2 &= 1140.7594 \rightarrow \sqrt{\text{both sides}} \\
\sqrt{a^2} &= \sqrt{1140.7594} \rightarrow \text{simplify} \\
a &\approx 33.8 \text{ units}
\end{aligned}$$

$$\begin{aligned}
\frac{\sin A}{a} &= \frac{\sin D}{d} \\
\frac{\sin A}{a} &= \frac{\sin D}{d} \rightarrow \angle A = 36^\circ,\ a = 33.8,\ d = 22 \\
\frac{\sin(36^\circ)}{33.8} &= \frac{\sin D}{22} \rightarrow \text{simplify} \\
\sin(36^\circ)(22) &= (33.8)\sin D \rightarrow \text{simplify} \\
(0.5878)(22) &= (33.8)(\sin D) \rightarrow \text{simplify} \\
12.9316 &= (33.8)\sin D \rightarrow \text{solve} \\
\frac{12.9316}{33.8} &= \frac{\cancel{(33.8)}\sin D}{\cancel{33.8}} \rightarrow \text{solve} \\
0.3826 &= \sin D \rightarrow \text{solve} \\
\sin^{-1}(0.3826) &= \sin^{-1}(\sin D) \\
22.5^\circ &\approx \angle D
\end{aligned}$$

This angle is from the horizontal.

b)



If $\angle CDA = 28^\circ$ then $\angle ABC = 28^\circ$

$\angle BCD = \angle BAD$

$$\begin{aligned}
\angle CDA + \angle ABC + \angle BCD + \angle BAD &= 360^\circ \\
2\angle CDA + 2\angle BCD &= 360^\circ \\
2(28^\circ) + 2\angle BCD &= 360^\circ \\
56^\circ + 2\angle BCD &= 360^\circ \\
2\angle BCD &= 360^\circ - 56^\circ \\
2\angle BCD &= 304^\circ \\
\frac{\cancel{2}\angle BCD}{\cancel{2}} &= \frac{304^\circ}{2} \\
\angle BCD &= 152^\circ
\end{aligned}$$

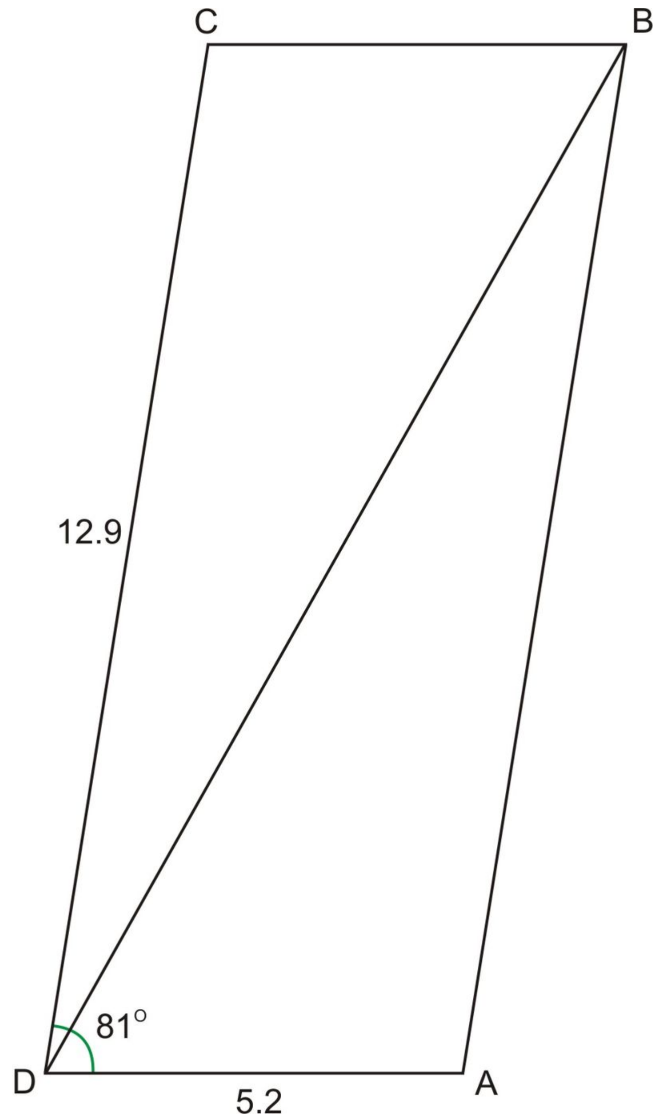
In $\triangle BAD$

$$
\begin{aligned}
a^2 &= b^2 + d^2 - 2bd\cos A \\
a^2 &= b^2 + d^2 - 2bd\cos A \to b = 19,\ d = 71,\ \angle A = 152^\circ \\
a^2 &= (19)^2 + (71)^2 - 2(19)(71)\cos(152^\circ) \to \text{simplify} \\
a^2 &= 7784.1926 \to \sqrt{\text{both sides}} \\
\sqrt{a^2} &= \sqrt{7784.1926} \to \text{simplify} \\
a &\approx 88.2 \text{ units}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\sin A}{a} &= \frac{\sin D}{d} \\
\frac{\sin A}{a} &= \frac{\sin D}{d} \to \angle A = 152^\circ,\ a = 88.2,\ d = 19 \\
\frac{\sin(152^\circ)}{88.2} &= \frac{\sin D}{19} \to \text{simplify} \\
\sin(152^\circ)(19) &= (88.2)\sin D \to \text{simplify} \\
(0.4695)(19) &= (88.2)\sin D \to \text{simplify} \\
(8.9205) &= (88.2)(\sin D) \to \text{solve} \\
\frac{8.9205}{88.2} &= \frac{\cancel{(88.2)}\sin D}{\cancel{88.2}} \to \text{solve} \\
0.1011 &= \sin D \to \text{solve} \\
\sin^{-1}(0.1011) &= \sin^{-1}(\sin D) \\
5.8^\circ &\approx \angle D
\end{aligned}
$$

This angle is from the horizontal.

c)



If $\angle CDA = 81^\circ$ then $\angle ABC = 81^\circ$

$\angle BCD = \angle BAD$

$$\begin{aligned}
\angle CDA + \angle ABC + \angle BCD + \angle BAD &= 360^\circ \\
2\angle CDA + 2\angle BCD &= 360^\circ \\
2(81^\circ) + 2\angle BCD &= 360^\circ \\
162^\circ + 2\angle BCD &= 360^\circ \\
2\angle BCD &= 360^\circ - 162^\circ \\
2\angle BCD &= 198^\circ \\
\frac{\cancel{2}\angle BCD}{\cancel{2}} &= \frac{198^\circ}{2} \\
\angle BCD &= 99^\circ
\end{aligned}$$

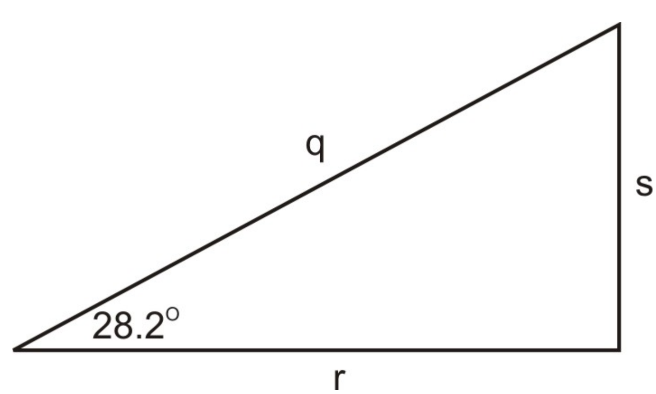
In $\triangle BAD$

$$
\begin{aligned}
a^2 &= b^2+d^2-2bd\cos A\\
a^2 &= b^2+d^2-2bd\cos A \rightarrow b=5.2,\ d=12.9,\ \angle A=99^\circ\\
a^2 &= (5.2)^2+(12.9)^2-2(5.2)(12.9)\cos(99^\circ) \rightarrow \text{simplify}\\
a^2 &= 214.4372 \rightarrow \sqrt{\text{both sides}}\\
\sqrt{a^2} &= \sqrt{214.4372} \rightarrow \text{simplify}\\
a &\approx 14.6 \text{ units}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\sin A}{a} &= \frac{\sin D}{d}\\
\frac{\sin A}{a} &= \frac{\sin D}{d} \rightarrow \angle A=99^\circ,\ a=14.6,\ d=5.2\\
\frac{\sin(99^\circ)}{14.6} &= \frac{\sin D}{5.2} \rightarrow \text{simplify}\\
\sin(99^\circ)(5.2) &= (14.6)\sin D \rightarrow \text{simplify}\\
(0.9877)(5.2) &= (14.6)(\sin D) \rightarrow \text{simplify}\\
5.1360 &= (14.6)\sin D \rightarrow \text{solve}\\
\frac{5.1360}{14.6} &= \frac{\cancel{(14.6)}\sin D}{\cancel{14.6}} \rightarrow \text{solve}\\
0.3518 &= \sin D \rightarrow \text{solve}\\
\sin^{-1}(0.3518) &= \sin^{-1}(\sin D)\\
20.6^\circ &\approx \angle D
\end{aligned}
$$

This angle is from the horizontal.

7. To determine the horizontal and vertical components, use the trigonometric ratio for cosine to calculate the horizontal component and the ratio for sine to calculate the vertical component.



$$\cos 28.2^\circ = \frac{|\vec{r}|}{|\vec{q}|} = \frac{r}{q} \qquad \sin 28.2^\circ = \frac{|\vec{s}|}{|\vec{q}|} = \frac{s}{q}$$

$$\cos 28.2^\circ = \frac{r}{12} \qquad \sin 28.2^\circ = \frac{s}{12}$$

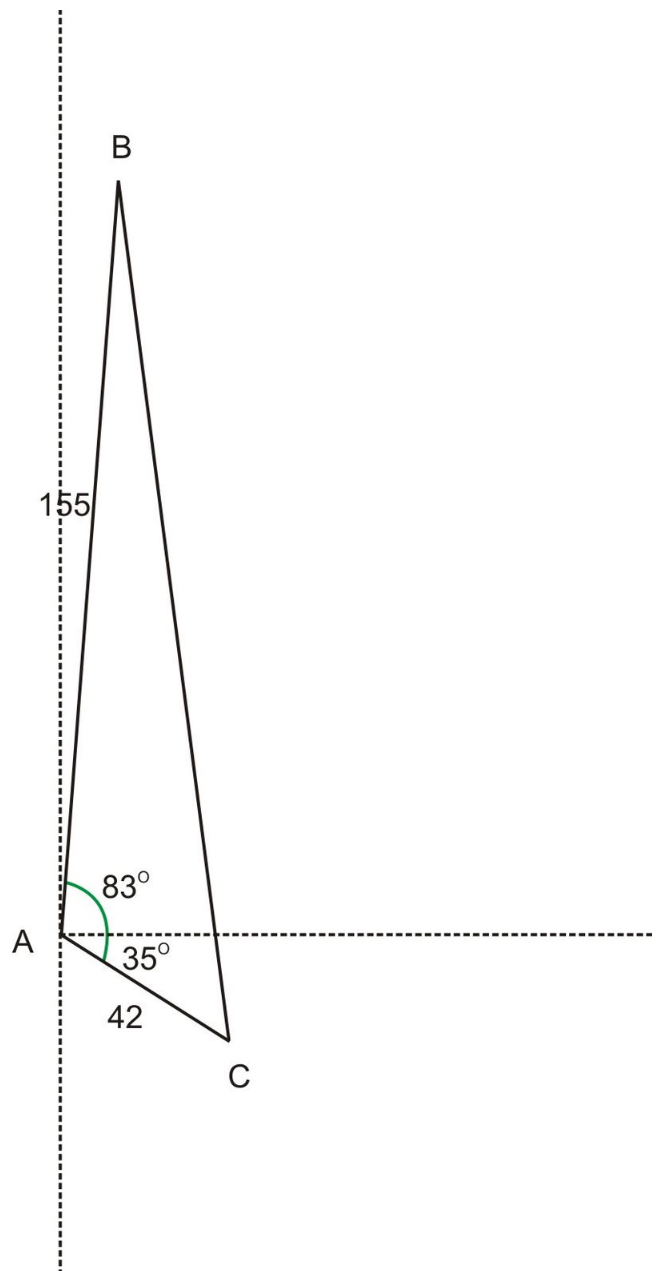
$$0.8813 = \frac{r}{12} \qquad 0.4726 = \frac{s}{12}$$

$$12(0.8813) = \cancel{12}\left(\frac{r}{\cancel{12}}\right) \qquad 12(0.4726) = \cancel{12}\left(\frac{s}{\cancel{12}}\right)$$

$$|10.6| \approx r(\text{horizontal}) \qquad |5.67| \approx s(\text{vertical})$$

$$10.6 \approx r(\text{horizontal}) \qquad 5.67 \approx s(\text{vertical})$$

8. To determine the heading of the plane, the Law of Cosines must be used to determine the magnitude of the plane and then the Law of Sines to calculate the heading.



$$
\begin{aligned}
a^2 &= b^2+c^2-2bc\cos A\\
a^2 &= b^2+c^2-2bc\cos A \to \angle A=118^\circ,\ b=42,\ c=155\\
a^2 &= (42)^2+(155)^2-2(42)(155)\cos(118^\circ) \to \text{simplify}\\
a^2 &= 31901.5198 \to \sqrt{\text{both sides}}\\
\sqrt{a^2} &= \sqrt{31901.5198} \to \text{simplify}\\
a &\approx 178.6 \text{ km/h}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\sin A}{a} &= \frac{\sin B}{b}\\
\frac{\sin A}{a} &= \frac{\sin B}{b} \to \angle A=118^\circ,\ a=178.6,\ b=42\\
\frac{\sin(118^\circ)}{178.6} &= \frac{\sin B}{42} \to \text{simplify}\\
\sin(118^\circ)(42) &= (178.6)\sin B \to \text{simplify}\\
(0.8829)(42) &= (178.6)(\sin B) \to \text{simplify}\\
37.0818 &= (178.6)\sin B \to \text{solve}\\
\frac{37.0818}{178.6} &= \frac{\cancel{(178.6)}\sin B}{\cancel{178.6}} \to \text{solve}\\
0.2076 &= \sin B \to \text{solve}\\
\sin^{-1}(0.2076) &= \sin^{-1}(\sin B)\\
12^\circ &\approx \angle B
\end{aligned}
$$

The heading is $12^\circ+83^\circ \approx 95^\circ$

9. The first step is to apply the Pythagorean to determine the speed the boat will travel with the current to cross the river and then use the trigonometric ratio for sine to calculate the angle at which the boat must travel.

$$
\begin{aligned}
(h)^2 &= (s_1)^2+(s_2)^2\\
(h)^2 &= (10)^2+(2)^2\\
(h)^2 &= 104\\
\sqrt{h^2} &= \sqrt{104}\\
10.2 \text{ mph} &\approx h
\end{aligned}
$$

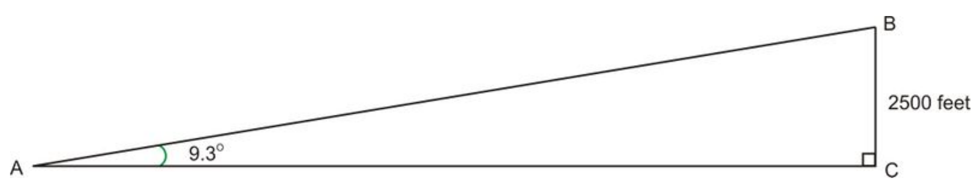
$$\sin A = \frac{\text{opp}}{\text{hyp}}$$
$$\sin A = \frac{\text{opp}}{\text{hyp}} \rightarrow \text{opp} = 2.00,\ \text{hyp} = 10.2$$
$$\sin A = \frac{2.00}{10.2} \rightarrow \text{simplify}$$
$$\sin A = 0.1961$$
$$\sin^{-1}(\sin A) = \sin^{-1}(0.1961)$$
$$\angle A \approx 11.3^\circ$$

10. If AB is any vector, then BA is a vector of the same magnitude but in the opposite direction. $AB + (-BA) = (0,0)$
.

Real-World Triangle Problem Solving

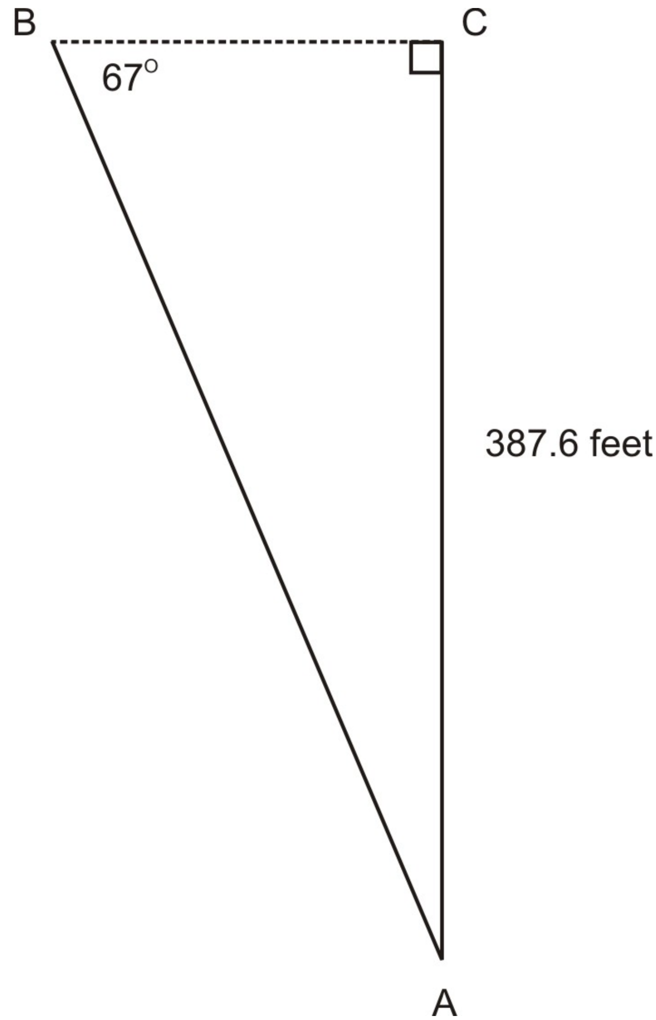
Review Exercises:

1. To determine the distance from the command post to a point on the ground directly below the helicopter, use the trigonometric ratio for tangent.



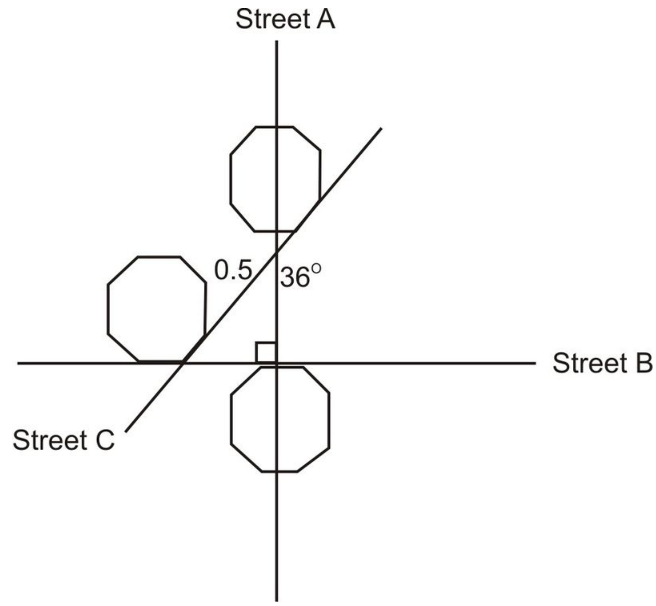
$$\tan A = \frac{\text{opp}}{\text{adj}}$$
$$\tan A = \frac{\text{opp}}{\text{adj}} \rightarrow \text{opp} = 2500,\ \text{adj} = x,\ \angle A = 9.3^\circ$$
$$\tan(9.3^\circ) = \frac{2500}{x} \rightarrow \text{simplify}$$
$$(0.1638) = \frac{2500}{x} \rightarrow \text{simplify}$$
$$(0.1638)(x) = (\cancel{x})\left(\frac{2500}{\cancel{x}}\right) \rightarrow \text{simplify}$$
$$(0.1638)(x) = 2500 \rightarrow \text{solve}$$
$$\frac{\cancel{(0.1638)}(x)}{\cancel{.1638}} = \frac{2500}{.1638} \rightarrow \text{solve}$$
$$x \approx 15262.5 \text{ feet}$$

2. To determine the distance across the canyon, use the trigonometric ratio for tangent.



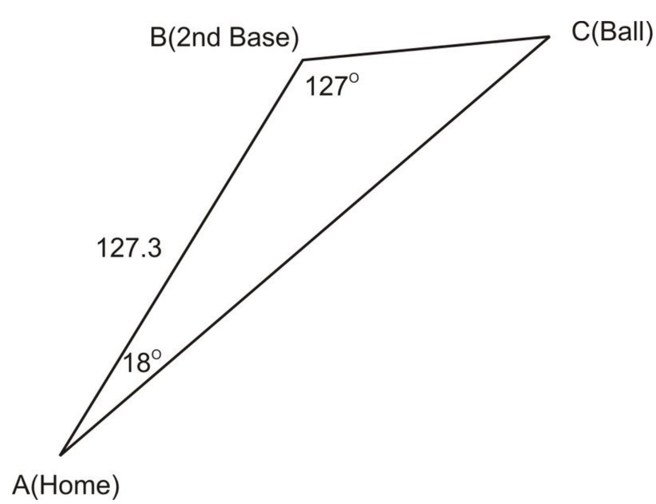
$$
\begin{aligned}
\tan B &= \frac{\text{opp}}{\text{adj}} \\
\tan B &= \frac{\text{opp}}{\text{adj}} \rightarrow \text{opp} = 387.6,\ \text{adj} = x,\ \angle B = 67^\circ \\
\tan(67^\circ) &= \frac{387.6}{x} \rightarrow \text{simplify} \\
(2.3559) &= \frac{387.6}{x} \rightarrow \text{simplify} \\
(2.3559)(x) &= (\cancel{x})\left(\frac{387.6}{\cancel{x}}\right) \rightarrow \text{simplify} \\
(2.3559)(x) &= 387.6 \rightarrow \text{solve} \\
\frac{\cancel{(2.3559)}(x)}{\cancel{2.3559}} &= \frac{387.6}{2.3559} \rightarrow \text{solve} \\
x &\approx 164.5 \text{ feet}
\end{aligned}
$$

3. To determine the distance between the stoplights on Street A, use the Trigonometric ratio for Sine.



$$
\begin{aligned}
\sin A &= \frac{\text{opp}}{\text{hyp}} \\
\sin A &= \frac{\text{opp}}{\text{hyp}} \rightarrow \text{opp} = x,\ \text{hyp} = 0.5,\ \angle A = 54^\circ \\
\sin(54^\circ) &= \frac{x}{0.5} \rightarrow \text{simplify} \\
(0.8090) &= \frac{x}{0.5} \rightarrow \text{simplify} \\
(0.8090)(0.5) &= \cancel{(0.5)}\left(\frac{x}{\cancel{0.5}}\right) \rightarrow \text{solve} \\
0.4 \text{ miles} &\approx x
\end{aligned}
$$

4. To determine the distance that the ball was shot and the distance of the second baseman from the ball, the Law of Sines and/or the Law of Cosines may be used.



$$\angle C = 180^\circ - (127^\circ + 18^\circ)$$
$$\angle C = 180^\circ - (145^\circ)$$
$$\angle C = 35^\circ$$

Distance the ball was hit

$$\begin{aligned}
\frac{c}{\sin C} &= \frac{b}{\sin B}\\
\frac{c}{\sin C} &= \frac{b}{\sin B} \to c = 127.3,\ \angle C = 35^\circ,\ \angle B = 127^\circ\\
\frac{127.3}{\sin(35^\circ)} &= \frac{b}{\sin(127^\circ)} \to \text{simplify}\\
\frac{127.3}{0.5734} &= \frac{b}{0.7986} \to \text{simplify}\\
127.3(0.7986) &= (0.5734)b \to \text{simplify}\\
101.6663 &= (0.5734)b \to \text{solve}\\
\frac{101.6663}{0.5734} &= \frac{\cancel{(0.5734)}b}{\cancel{0.5734}} \to \text{solve}\\
177.2 \text{ feet} &\approx b
\end{aligned}$$

Distance the ball is from the second baseman

$$\begin{aligned}
\frac{c}{\sin C} &= \frac{a}{\sin A}\\
\frac{c}{\sin C} &= \frac{a}{\sin A} \to c = 127.3,\ \angle C = 35^\circ,\ \angle A = 18^\circ\\
\frac{127.3}{\sin(35^\circ)} &= \frac{a}{\sin(18^\circ)} \to \text{simplify}\\
\frac{127.3}{0.5734} &= \frac{a}{0.3090} \to \text{simplify}\\
127.3(0.3090) &= (0.5734)a \to \text{simplify}\\
39.3379 &= (0.5734)a \to \text{solve}\\
\frac{39.3379}{0.5734} &= \frac{\cancel{(0.5734)}b}{\cancel{0.5734}} \to \text{solve}\\
68.6 \text{ feet} &\approx a
\end{aligned}$$

5. There is not enough information given in this question to answer it.
6. To determine the distance from the Tower to Target 2, the Law of Cosines must be used.

$$\begin{aligned}
\text{Target } 1 &= A\\
\text{Target } 2 &= B\\
\text{Tower } &= C
\end{aligned}$$

$$\begin{aligned}
a^2 &= b^2+c^2-2bc\cos A\\
a^2 &= b^2+c^2-2bc\cos A \rightarrow \angle A = 67.2^\circ,\ b=18,\ c=37\\
a^2 &= (18)^2+(37)^2-2(18)(37)\cos(67.2^\circ) \rightarrow \text{simplify}\\
a^2 &= 1176.8292 \rightarrow \sqrt{\text{both sides}}\\
\sqrt{a^2} &= \sqrt{1176.8292}\\
a &\approx 34.3 \text{ miles}
\end{aligned}$$

The sensor will not be able to detect the second target. Target 2 is out of range by approximately 4.3 miles .

7. To determine the number of bacteria, the area of the lake must be calculated. The Law of Cosines must be used to determine the measure of one of the angles of the triangle. Then the formula $K=\frac{1}{2}bc\sin A$ can be used to calculate the area of the lake.

$$\begin{aligned}
\text{Dock } 1 &= A\\
\text{Dock } 2 &= B\\
\text{Dock } 3 &= C
\end{aligned}$$

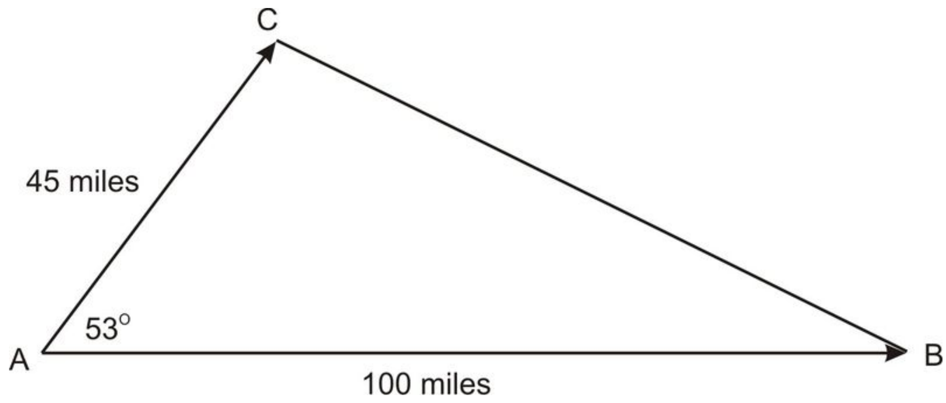
$$\begin{aligned}
\cos A &= \frac{b^2+c^2-a^2}{2bc}\\
\cos A &= \frac{b^2+c^2-a^2}{2bc} \rightarrow a=587,\ b=396,\ c=247\\
\cos A &= \frac{(396)^2+(247)^2-(587)^2}{2(396)(247)} \rightarrow \text{simplify}\\
\cos A &= \frac{-126744}{195624} \rightarrow \text{divide}\\
\cos A &= -0.6479\\
\cos^{-1}(\cos A) &= \cos^{-1}(-0.6479)\\
\angle A &\approx 130.4^\circ
\end{aligned}$$

Area of lake:

$$\begin{aligned}
K &= \frac{1}{2}bc\sin A\\
K &= \frac{1}{2}bc\sin A \rightarrow b=396,\ c=247,\ \angle A = 130.4^\circ\\
K &= \frac{1}{2}(396)(247)\sin(130.4^\circ) \rightarrow \text{simplify}\\
K &= 37243.8 \text{ ft}^2
\end{aligned}$$

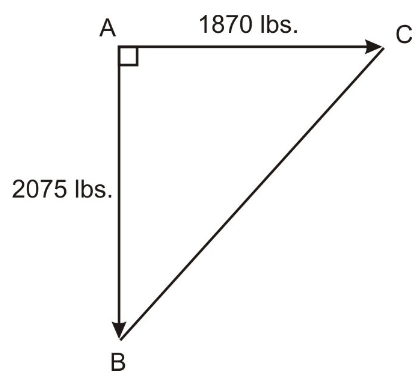
The numbers of bacteria that are living on the surface of the lake are $37243.8(5.2\times10^{13})\approx 1.94\times10^{18}$

8. A direction of 37° east of north is an angle of 53° with the horizontal. This must be considered when drawing the diagram to represent the problem and when calculating the distance from Tower B to the fire. This distance can be calculated by using the Law of Cosines.



$$
\begin{aligned}
a^2 &= b^2+c^2-2bc\cos A\\
a^2 &= b^2+c^2-2bc\cos A \rightarrow \angle A = 53^\circ,\ b=45,\ c=100\\
a^2 &= (45)^2+(100)^2-2(45)(100)\cos(53^\circ) \rightarrow \text{simplify.}\\
a^2 &= 6608.6648 \rightarrow \sqrt{\text{both sides}}\\
\sqrt{a^2} &= \sqrt{6608.6648}\\
a &\approx 81.3 \text{ miles}
\end{aligned}
$$

9. The two forces are acting at right angles to each other due to the direction of the forces. The Pythagorean Theorem can be used to determine the magnitude of the resultant on the footing and the tangent function may be used to calculate the direction of the resultant.



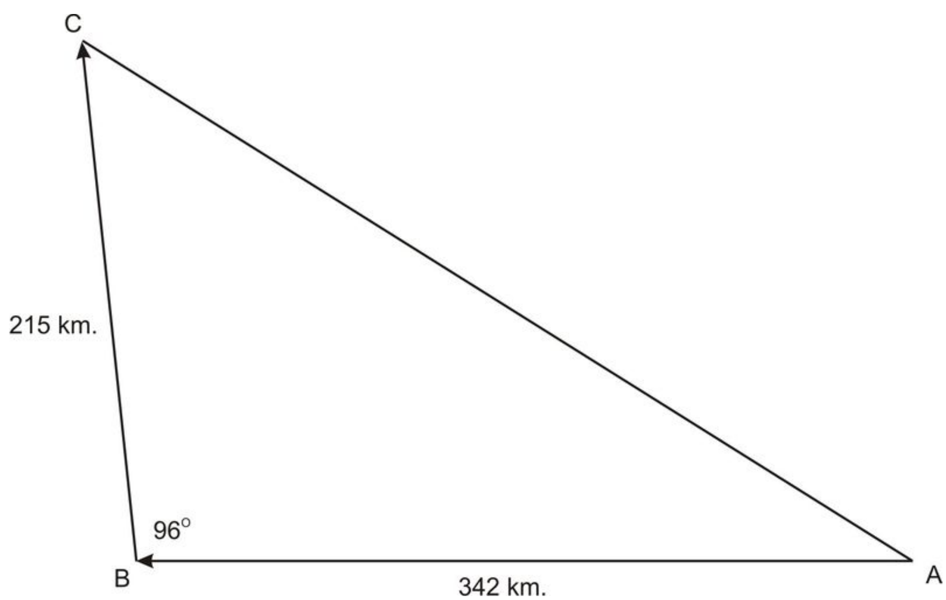
a)

$$
\begin{aligned}
(h)^2 &= (s_1)^2+(s_2)^2\\
(h)^2 &= (1870)^2+(2075)^2\\
(h)^2 &= 7802525\\
\sqrt{h^2} &= \sqrt{7802525}\\
2793.3 \text{ lbs.} &\approx h
\end{aligned}
$$

b)

$$\begin{aligned}
\tan C &= \frac{\text{opp}}{\text{adj}}\\
\tan C &= \frac{\text{opp}}{\text{adj}} \rightarrow \text{opp} = 2075,\ \text{adj} = 1870\\
\tan C &= 1.1096 \rightarrow \text{simplify}\\
\tan^{-1}(\tan C) &= \tan^{-1}(1.1096)\\
\angle C &\approx 48^\circ
\end{aligned}$$

10. A heading of 118° is an angle of 62° with the horizontal. A second heading of 34° will result in an angle of $62^\circ + 34^\circ = 96^\circ$.



$$\begin{aligned}
b^2 &= a^2 + c^2 - 2ac\cos B\\
b^2 &= a^2 + c^2 - 2ac\cos C \rightarrow \angle C = 96^\circ,\ a = 215,\ c = 342\\
b^2 &= (215)^2 + (342)^2 - 2(215)(342)\cos(96^\circ) \rightarrow \text{simplify.}\\
b^2 &= 178560.9558 \rightarrow \sqrt{\text{both sides}}\\
\sqrt{b^2} &= \sqrt{178560.9558}\\
b &\approx 422.6\text{ km}
\end{aligned}$$

$$\begin{aligned}
\frac{\sin B}{b} &= \frac{\sin C}{c}\\
\frac{\sin B}{b} &= \frac{\sin C}{c} \rightarrow \angle B = 96^\circ,\ b = 422.6,\ c = 342\\
\frac{\sin(96^\circ)}{422.6} &= \frac{\sin C}{342} \rightarrow \text{simplify}\\
\sin(96^\circ)(342) &= (422.6)\sin C \rightarrow \text{simplify}\\
(0.9945)(342) &= (422.6)\sin C \rightarrow \text{simplify}\\
340.119 &= (422.6)\sin C \rightarrow \text{solve}\\
\frac{340.119}{422.6} &= \frac{\cancel{(422.6)}\sin D}{\cancel{422.6}} \rightarrow \text{solve}\\
0.8048 &= \sin C \rightarrow \text{solve}\\
\sin^{-1}(0.8048) &= \sin^{-1}(\sin C)\\
53.6^\circ &\approx \angle C
\end{aligned}$$

The heading is $53.6^\circ + 34^\circ \approx 87.6^\circ$

CHAPTER 6 Polar Equations and Complex Numbers - Solution Key

CHAPTER OUTLINE

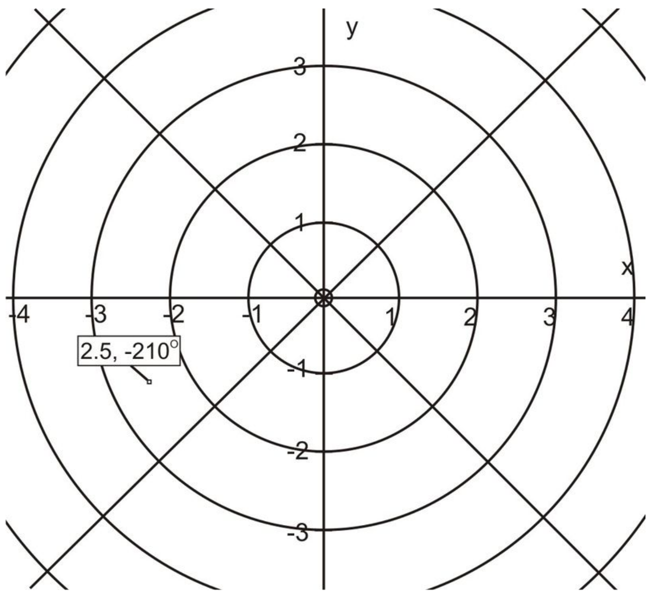
6.1 Polar Equations and Complex Numbers

Polar Coordinates

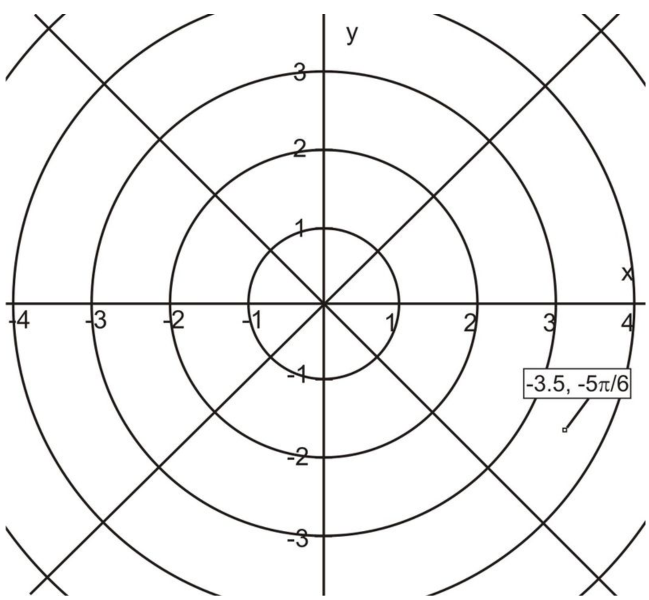
Review Exercises

1. To plot these points using computer software, choose **polar** as the grid. Then enter the coordinates.

a)



b)



2. To determine four pair of polar coordinates to represent the point $A\left(-4,\frac{\pi}{4}\right)$, use the formula $(r,\theta+2\pi k)$ and choose different values for k . Then use the formula $(r,\theta+[2k+1]\pi)$ and again choose different values for k .

$$\text{Using } (r,\theta+2\pi k) \text{ and } k=-1$$
$$(r,\theta+2\pi k)\rightarrow r=-4,\theta=\frac{\pi}{4},k=-1$$
$$\left(-4,\frac{\pi}{4}+2\pi(-1)\right)\rightarrow \text{simplify}$$
$$\left(-4,\frac{\pi}{4}-2\pi\right)\rightarrow \text{common deno min ator}$$
$$\left(-4,\frac{\pi}{4}-\frac{8\pi}{4}\right)\rightarrow \text{simplify}$$
$$\left(-4,-\frac{7\pi}{4}\right)$$

$$\text{Using } (r,\theta+2\pi k) \text{ and } k=-\frac{1}{2}$$
$$(r,\theta+2\pi k)\rightarrow r=-4,\theta=\frac{\pi}{4},k=-\frac{1}{2}$$
$$\left(-4,\frac{\pi}{4}+\cancel{2}\pi\left(-\frac{1}{\cancel{2}}\right)\right)\rightarrow \text{simplify}$$
$$\left(-4,\frac{\pi}{4}+(-1)\pi\right)\rightarrow \text{simplify}$$
$$\left(-4,\frac{\pi}{4}-\pi\right)\rightarrow \text{common deno min ator}$$
$$\left(-4,\frac{\pi}{4}-\frac{4\pi}{4}\right)\rightarrow \text{simplify}$$
$$\left(-4,-\frac{3\pi}{4}\right)$$

$$\text{Using } (r,\theta+[2k+1]\pi) \text{ and } k=-1$$
$$(r,\theta+[2k+1]\pi)\rightarrow r=4,\theta=\frac{\pi}{4},k=-1$$
$$\left(4,\frac{\pi}{4}+[2(-1)+1]\pi\right)\rightarrow \text{simplify}$$
$$\left(4,\frac{\pi}{4}+[-2+1]\pi\right)\rightarrow \text{simplify}$$
$$\left(4,\frac{\pi}{4}+[-2+1]\pi\right)\rightarrow \text{simplify}$$
$$\left(4,\frac{\pi}{4}+[-1]\pi\right)\rightarrow \text{simplify}$$
$$\left(4,\frac{\pi}{4}-\pi\right)\rightarrow \text{common deno min ator}$$
$$\left(4,\frac{\pi}{4}-\frac{4\pi}{4}\right)\rightarrow \text{simplify}$$
$$\left(4,-\frac{3\pi}{4}\right)$$

$$\text{Using } (r,\theta+[2k+1]\pi) \text{ and } k=0$$
$$(r,\theta+[2k+1]\pi)\rightarrow r=4,\theta=\frac{\pi}{4},k=0$$
$$\left(4,\frac{\pi}{4}+[2(0)+1]\pi\right)\rightarrow \text{simplify}$$
$$\left(4,\frac{\pi}{4}+[0+1]\pi\right)\rightarrow \text{simplify}$$
$$\left(4,\frac{\pi}{4}+\pi\right)\rightarrow \text{common deno min ator}$$
$$\left(4,\frac{\pi}{4}+\frac{4\pi}{4}\right)\rightarrow \text{simplify}$$
$$\left(4,\frac{5\pi}{4}\right)$$

First Pair $\rightarrow\left(-4,-\frac{7\pi}{4}\right)$

Second Pair $\rightarrow\left(-4,-\frac{3\pi}{4}\right)$

Third Pair $\rightarrow\left(4,-\frac{3\pi}{4}\right)$

Fourth Pair $\rightarrow\left(4,\frac{5\pi}{4}\right)$

3. To calculate the distance between the points use the distance formula for polar coordinates which is a form of the Law of Cosines. Use the formula $p_1p_2=\sqrt{r_1^2+r_2^2-2r_1r_2\cos(\theta_2-\theta_1)}$ and the coordinates $\begin{pmatrix} r_1,\theta_1 \\ 1,30^\circ \end{pmatrix}$ and $\begin{pmatrix} r_2,\theta_2 \\ 6,135^\circ \end{pmatrix}$.

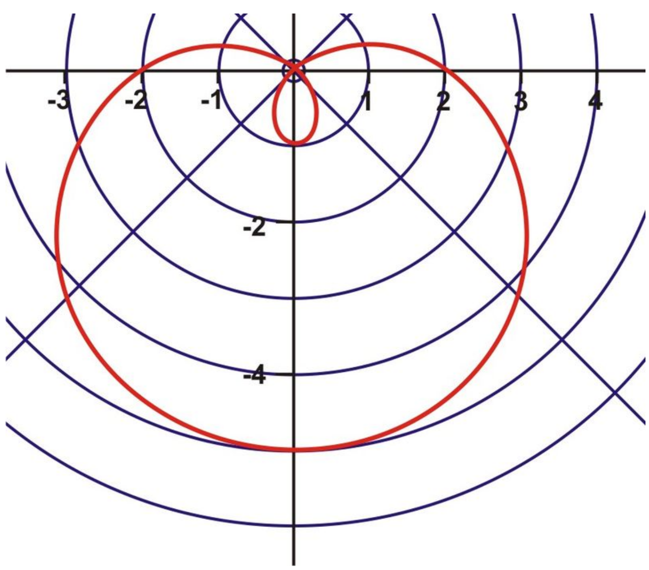
$$p_1p_2 = \sqrt{r_1^2+r_2^2-2r_1r_2\ \cos(\theta_2-\theta_1)}$$
$$p_1p_2 = \sqrt{r_1^2+r_2^2-2r_1r_2\ \cos(\theta_2-\theta_1)} \rightarrow r_1=1, r_2=6, \theta_1=30^\circ, \theta_2=135^\circ$$
$$p_1p_2 = \sqrt{(1)_1^2+(6)^2-2(1)(6)\cos(135^\circ-30^\circ)} \rightarrow \text{simplify}$$
$$p_1p_2 = \sqrt{40.1058} \rightarrow \text{simplify}$$
$$p_1p_2 \approx 6.33 \text{ units}$$

Sinusoids of one Revolution (e.g. limaçons, cardioids)

Review Exercises

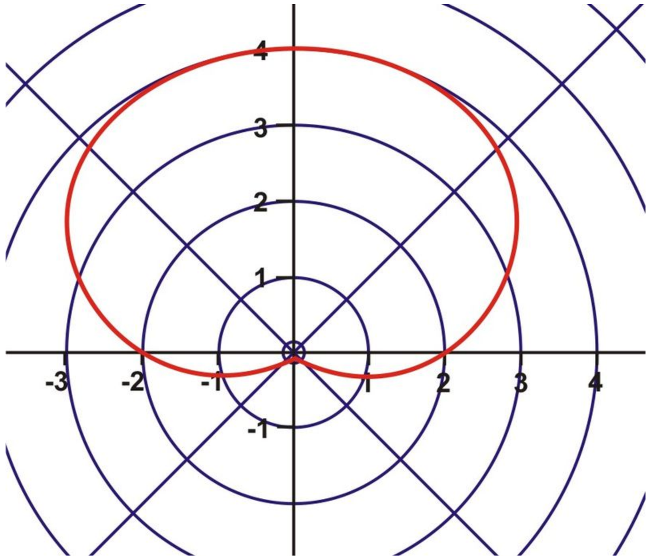
1.

a)



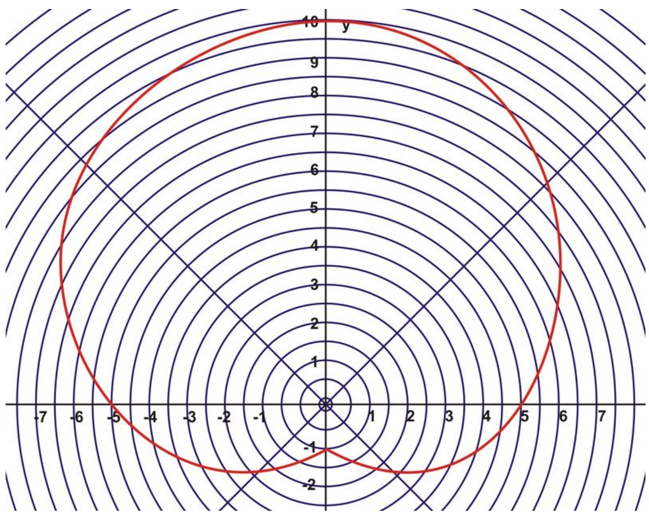
A limaçon with an inner loop

b)



A cardioid

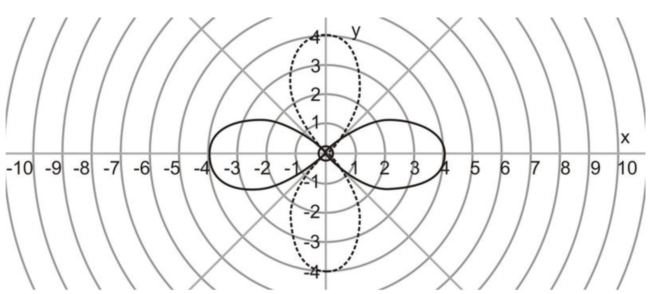
c)



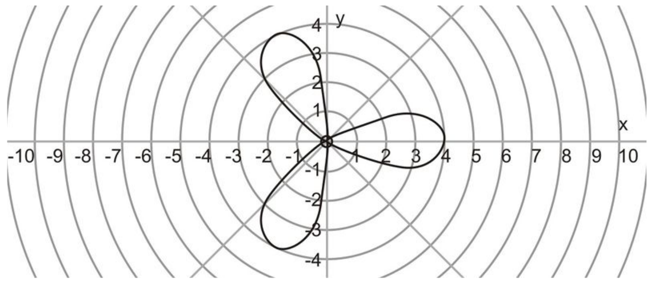
A dimpled limaçon

2. For the equation $r = 4\cos\ 2\theta$ *such that* $0^\circ \le \theta \le 360^\circ$, create a table of values and sketch the graph. Repeat the process for $r = 4\cos\ 3\theta$ *such that* $0^\circ \le \theta \le 360^\circ$

θ	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°	360°
$4\cos\ 2\theta$	4	2	-2	-4	-2	2	4	2	-2	-4	-2	2	4



θ	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°	360°
$4\cos\ 3\theta$	4	0	-4	0	4	0	-4	0	4	0	-4	0	4



The number n has an affect on the number of petals on the rose. The first graph, $r = 4\cos\ 2\theta$ *such that* $0^\circ \le \theta \le 360^\circ$ the rose has four petals on it. In this case, n is an even, positive integer and the rose has an even number of petals. The second graph, $r = 4\cos\ 3\theta$ *such that* $0^\circ \le \theta \le 360^\circ$ the rose has three petals on it. In this case, n is an odd, positive integer and the rose has an odd number of petals.

Graphs of Polar Equations

Review Exercises

1. To determine the rectangular coordinates of polar coordinates means to express the given point as (x, y) . To do this use the formula $x = r\cos\ \theta$ to determine the $x-$ coordinate and the formula $y = r\sin\ \theta$ to determine the $y-$ coordinate.

a) $A\left(-4, \frac{5\pi}{4}\right)$

$$x = r\cos\ \theta$$
$$x = r\cos\ \theta \to r = -4, \theta = \frac{5\pi}{4}$$
$$x = (-4)\cos\left(\frac{5\pi}{4}\right) \to \text{simplify}$$
$$x = -4\cos\left(\frac{5\pi}{4}\right) \to \cos\frac{5\pi}{4} = -\frac{\sqrt{2}}{2}$$
$$x = -4\left(-\frac{\sqrt{2}}{2}\right) \to \text{simplify}$$
$$x = -2\cancel{4}\left(-\frac{\sqrt{2}}{\cancel{2}}\right) \to \text{solve}$$
$$x = 2\sqrt{2}$$

$$y = r\sin\theta$$
$$y = r\sin\ \theta \to r = -4, \theta = \frac{5\pi}{4}$$
$$y = (-4)\sin\left(\frac{5\pi}{4}\right) \to \text{simplify}$$
$$y = -4\sin\left(\frac{5\pi}{4}\right) \to \sin\frac{5\pi}{4} = -\frac{\sqrt{2}}{2}$$
$$y = -4\left(-\frac{\sqrt{2}}{2}\right) \to \text{simplify}$$
$$y = -2\cancel{4}\left(-\frac{\sqrt{2}}{\cancel{2}}\right) \to \text{solve}$$
$$y = 2\sqrt{2}$$

$$A\left(-4, \frac{5\pi}{4}\right) = (2\sqrt{2}, 2\sqrt{2})$$

b) $B(-3, 135^\circ)$

$$x = r\cos\ \theta \rightarrow r = -3, \theta = 135^\circ$$
$$x = (-3)\cos(135^\circ) \rightarrow \text{simplify}$$
$$x = -3\cos 135^\circ \rightarrow \cos 135^\circ = -\frac{\sqrt{2}}{2}$$
$$x = -3\left(-\frac{\sqrt{2}}{2}\right) \rightarrow \text{simplify}$$
$$x = \frac{3\sqrt{2}}{2}$$

$$y = r\sin\ \theta \rightarrow r = -3, \theta = 135^\circ$$
$$y = (-3)\sin(135^\circ) \rightarrow \text{simplify}$$
$$y = -3\sin(135^\circ) \rightarrow \sin 135^\circ = -\frac{\sqrt{2}}{2}$$
$$y = -3\left(\frac{\sqrt{2}}{2}\right) \rightarrow \text{simplify}$$
$$y = -\frac{3\sqrt{2}}{\cancel{2}}$$

$$B(-3, 135^\circ) = \left(\frac{3\sqrt{2}}{2}, \frac{3\sqrt{2}}{2}\right)$$

c) $C\left(5, \frac{2\pi}{3}\right)$

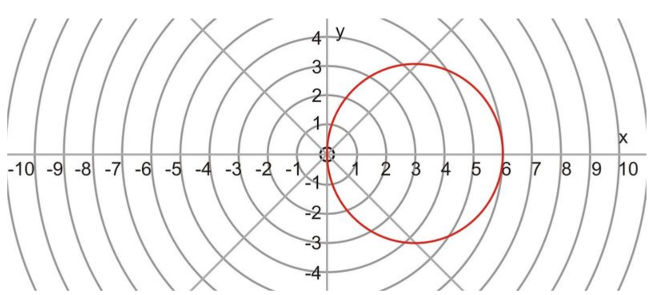
$$x = r\cos\ \theta$$
$$x = r\cos\ \theta \rightarrow r = 5, \theta = \frac{2\pi}{3}$$
$$x = (5)\cos\left(\frac{2\pi}{3}\right) \rightarrow \text{simplify}$$
$$x = (5)\cos\left(\frac{2\pi}{3}\right) \rightarrow \cos\frac{2\pi}{3} = -\frac{1}{2}$$
$$x = (5)\left(-\frac{1}{2}\right) \rightarrow \text{simplify}$$
$$x = -\frac{5}{2} \rightarrow \text{solve}$$
$$x = -2.5$$

$$y = r\sin\ \theta$$
$$y = r\sin\ \theta \rightarrow r = 5, \theta = \frac{2\pi}{3}$$
$$y = (5)\sin\left(\frac{2\pi}{3}\right) \rightarrow \text{simplify}$$
$$y = (5)\sin\left(\frac{2\pi}{3}\right) \rightarrow \sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2}$$
$$y = 5\left(\frac{\sqrt{3}}{2}\right) \rightarrow \text{simplify}$$
$$y = \frac{5\sqrt{3}}{2}$$

$$C\left(5, \frac{2\pi}{3}\right) = \left(-2.5, \frac{5\sqrt{3}}{2}\right)$$

2. $r = 6\cos\ \theta$

The following graph represents a circle with its center at $(3, 0)$ and a radius of 3 units .



$$r = 6\cos\ \theta$$
$$r^2 = 6\cos\ \theta$$
$$x^2+y^2 = 6\cos\ \theta \rightarrow \text{let} x = \cos\ \theta$$
$$x^2+y^2 = 6x \rightarrow \text{simplify}$$
$$x^2+y^2-6x = 6x-6x \rightarrow \text{simplify}$$
$$(x^2-6x)+y^2 = 0 \rightarrow \text{complete the square}$$
$$(x^2-6x+9)+y^2 = 0+9 \rightarrow \text{write as a perfect square trinomial}$$
$$(x-3)^2+y^2 = 9$$
$$(x-h)^2+(y-k)^2 = r^2 \rightarrow \text{general formula}$$
$$(x-3)^2+(y-0)^2 = 3^2$$

Rectangular to Polar

Review Exercises

1. To write rectangular coordinates in polar form, use the formula $r = \sqrt{x^2+y^2}$ to determine the value of r and the formula $\theta = \text{Arc}\ \tan\frac{y}{x}+\pi\ for\ x<0$ or the formula $\theta = \text{Arc}\ \tan\frac{y}{x}\ for\ x>0$ to calculate the value of θ .

a) $A(-2,5)$. This point is located in the 2nd quadrant and $x<0$.

$$r = \sqrt{x^2+y^2}$$
$$r = \sqrt{x^2+y^2} \rightarrow x=-2, y=5$$
$$r = \sqrt{(-2)^2+(5)^2} \rightarrow \text{simplify}$$
$$r = \sqrt{29} \rightarrow \text{simplify}$$
$$r \approx 5.39$$

$$\theta = \text{Arc}\ \tan\frac{y}{x}+\pi \text{for}\ x<0$$
$$\theta = \text{Arc}\ \tan\frac{y}{x}+\pi \rightarrow x=-2, y=5$$
$$\theta = \text{Arc}\ \tan\frac{5}{-2}+\pi \rightarrow \text{simplify}$$
$$\theta = \tan^{-1}(-2.5)+\pi \rightarrow \text{simplify}$$
$$\theta = -1.1903+\pi \rightarrow \text{simplify}$$
$$\theta \approx 1.95$$

$$A(-2,5) = (5.39, 1.95)$$

b) $B(5,-4)$. This point is located in the 4th quadrant and $x>0$.

$$r = \sqrt{x^2+y^2}$$
$$r = \sqrt{x^2+y^2} \rightarrow x=5, y=-4$$
$$r = \sqrt{(5)^2+(-4)^2} \rightarrow \text{simplify}$$
$$r = \sqrt{41} \rightarrow \text{simplify}$$
$$r \approx 6.40$$

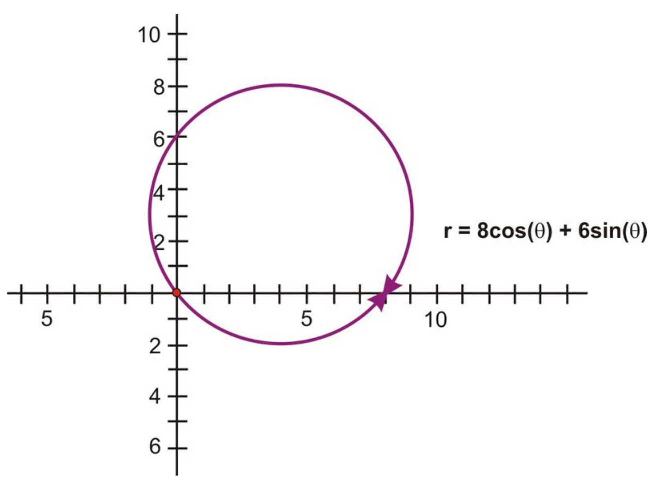
$$\theta = \text{Arc}\ \tan\frac{y}{x}\text{for}\ x<0$$
$$\theta = \text{Arc}\ \tan\frac{y}{x} \rightarrow x=5, y=-4$$
$$\theta = \text{Arc}\ \tan\frac{-4}{5} \rightarrow \text{simplify}$$
$$\theta = \tan^{-1}(-0.8) \rightarrow \text{simplify}$$
$$\theta = -0.67$$

$$B(5,-4) = (6.40, -0.67)$$

2. To write the equation $(x-4)^2+(y-3)^2 = 25$, expand the equation in terms of x and y . Then replace x with the expression $r\cos\ \theta$ and y with $r\sin\ \theta$.

$$
\begin{aligned}
(x-4)^2+(y-3)^2 &= 25\\
(x-4)^2+(y-3)^2 &= 25 \rightarrow \text{exp and}\\
(x^2-8x+16)+(y^2-6y+9) &= 25 \rightarrow \text{simplify}\\
x^2-8x+y^2-6y+25 &= 25 \rightarrow \text{simplify}\\
x^2-8x+y^2-6y+25-25 &= 25-25 \rightarrow \text{simplify}\\
x^2-8x+y^2-6y &= 0 \rightarrow x = r\cos\ \theta, y = r\sin\ \theta, r = x^2+y^2\\
r^2-8(r\cos\ \theta)-6(r\sin\ \theta) &= 0 \rightarrow \text{simplify}\\
r^2-8r\cos\ \theta-6r\sin\ \theta &= 0 \rightarrow \text{common factor}\\
r(r-8\cos\ \theta-6\sin\ \theta) &= 0 \rightarrow \text{solve}\\
r = 0 \text{ or } r-8\cos\ \theta-6\sin\ \theta &= 0
\end{aligned}
$$

The graph of $r = 0$ is a single point – the origin. The graph of $r-8\cos\ \theta-6\sin\ \theta = 0$ contains this single point. The polar form of $(x-4)^2+(y-3)^2 = 25$ as a single equation is $r = 8\cos\ \theta+6\sin\ \theta$ and the graph is



The graph was drawn on a polar grid and then the grid was deleted so as to reveal a clear view of the shape of the graph - a circle with its center at $(4,3)$ and a radius of 5 units . The circumference of the circle passes through the origin.

Polar Equations and Complex Numbers

Review Exercises

1. To prove that the equation represents a parabola, write the equation in standard form. Then determine the vertex (h,k) , the focus $(h+p,k)$ and the directrix $(x = h-p)$.

$$\begin{aligned}
y^2-4y-8x+20&=0\\
y^2-4y-8x+20+8x-20&=8x-20\rightarrow \text{simplify}\\
y^2-4y&=8x-20\rightarrow \text{complete the square}\\
y^2-4y+4&=8x-20+4\rightarrow \text{simplify}\\
y^2-4y+4&=8x-16\rightarrow \text{perfect square binomial}\\
(y-2)^2&=8x-16\rightarrow \text{common factor}\\
(y-2)^2&=8(x-2)
\end{aligned}$$

The equation is in standard form.

The **vertex** (h,k) is $(2,2)$.

$$4p=8\rightarrow\frac{\cancel{4}p}{\cancel{4}}=\frac{8}{4}\rightarrow p=2.$$

Therefore the **focus** is $(h+p,k)$ which equals $(2+2,2)\rightarrow(4,2)$.

The directrix, $(x=h-p)$ is $(x=2-2)\rightarrow x=0$.

2. To determine the center (h,k) , the vertices $(h\pm a,k)$, foci and the eccentricity $\left(\frac{c}{a}\right)$ of the ellipse, express the equation in standard form.

$$\begin{aligned}
9x^2+16y^2+54x-32y-47&=0\\
9x^2+16y^2+54x-32y-47+47&=0+47\rightarrow \text{simplify}\\
9x^2+16y^2+54x-32y&=47\rightarrow \text{common factor}\\
9x^2+54x+16y^2-32y&=47\rightarrow \text{common factor}\\
9(x^2+6x)+16(y^2-2y)&=47\rightarrow \text{complete the square}\\
9(x^2+6x+9)+16(y^2-2y+1)&=47\rightarrow \text{add to right side}\\
9(x^2+6x+9)+16(y^2-2y+1)&=47+81+16\rightarrow \text{simplify}\\
9(x^2+6x+9)+16(y^2-2y+1)&=144\rightarrow \text{perfect square binomial}\\
9(x+3)^2+16(y-1)^2&=144\rightarrow\div(144)\\
\frac{9(x+3)^2}{144}+\frac{16(y-1)^2}{144}&=\frac{144}{144}\rightarrow \text{simplify}\\
\frac{\cancel{9}(x+3)^2}{\cancel{144}(16)}+\frac{\cancel{16}(y-1)^2}{\cancel{144}(9)}&=1\rightarrow \text{simplify}\\
\frac{(x+3)^2}{4^2}+\frac{(y-1)^2}{3^3}&=1\\
\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}&=1\rightarrow \text{s tan dard form}
\end{aligned}$$

The **centre** is $(h,k)\rightarrow(-3,1)$.

The vertices are $(h\pm a,k)$ and $a=4$. Thus the **vertices** are $(-3\pm 4,1)\rightarrow(1,1)$ *and* $(-7,1)$.

The foci are $(h\pm a,k)$ and $c=\sqrt{a^2-b^2}\rightarrow c=\sqrt{4^2-3^2}\rightarrow\sqrt{16-9}=\sqrt{7}$ The **foci** are $(-3\pm\sqrt{7},1)\approx(-5.65,1)$ *and* $(-0.35,1)$.

The **eccentricity** $\left(\frac{c}{a}\right)$ is $\frac{\sqrt{7}}{4} \approx 0.66$.

3. To determine the eccentricity, the type of conic and the directrix, use the general formula $r = \frac{\text{de}}{1 - e\cos\ \theta}$.

$$
\begin{aligned}
r &= \frac{\text{de}}{1 - e\cos\ \theta} \\
r &= \frac{2}{4 - \cos\ \theta} \rightarrow \div(4) \\
r &= \frac{\frac{2}{4}}{\frac{4}{4} - \frac{1}{4}\cos\theta} \rightarrow \text{simplify} \\
r &= \frac{0.5}{1 - 0.25\cos\theta}
\end{aligned}
$$

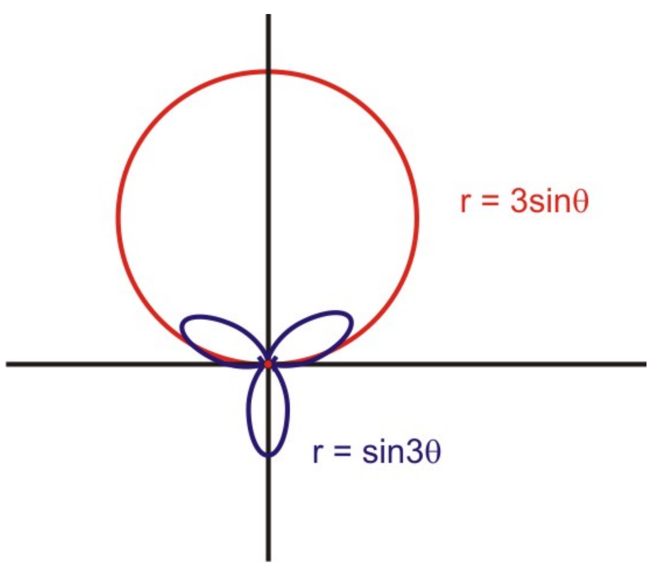
If $0 < e < 1$, the graph will be an ellipse. The eccentricity is 0.25 so the **conic** is an ellipse. The numerator $\text{de} = 0.5$. Therefore the directrix is $\frac{\text{de}}{e} \rightarrow \frac{0.5}{0.25} = 2$. The directrix is $x = -2$.

Graph and Calculate Intersections of Polar Curves

Review Exercises

1. To determine the points of intersection of the graphs, hide the grid when the graph has been completed. This makes it easier to determine the intersection. Then, solve the equations for each graph.

a) $r = \sin(3\theta)$ *and* $r = 3\sin\ \theta$



There appears to be one point of intersection – the origin.

Let $r = 0$

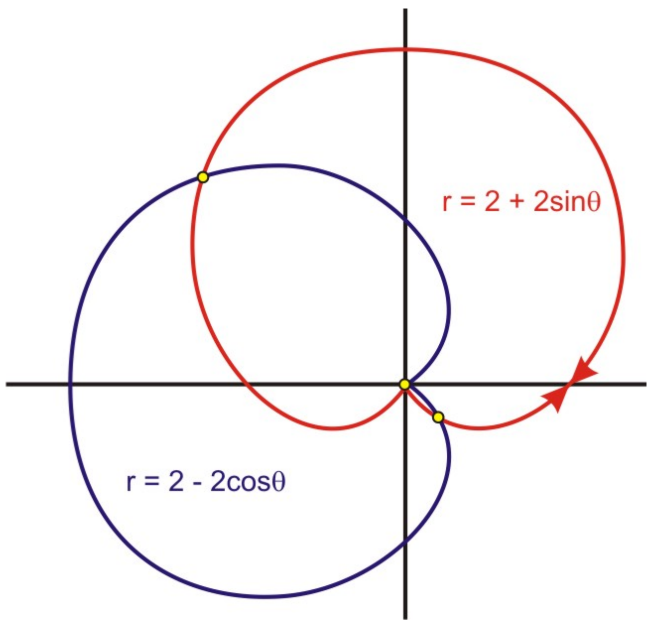
$$\begin{aligned} r\sin(3\theta) \\ 0 &= \sin\ 3\theta \\ \sin^{-1}(0) &= \sin^{-1}(\sin\ \theta) \\ \sin^{-1}(0) &= \sin^{-1}(\sin\ \theta) \\ 0 &= \theta \end{aligned}$$

$$\begin{aligned} r &= 3\sin\ \theta \\ 0 &= 3\sin\ \theta \\ \frac{0}{3} &= \frac{3\sin\ \theta}{3} \\ 0 &= \sin\ \theta \\ \sin^{-1}(0) &= \sin^{-1}(\sin\ \theta) \\ 0 &= \theta \end{aligned}$$

To accommodate 3θ , multiplying does not change the value of θ .

The point of intersection is $(0,0) \rightarrow r = 0, \theta = 0$

b) Plot the graphs of $r = 2 + 2\sin\ \theta$ and $r = 2 - 2\cos\ \theta$.



There appears to be three points of intersection.

One point of intersection seems to be the origin $(0,0)$.

Let $\theta = 0$

$$\begin{aligned} r &= 2 + 2\sin\ \theta \\ r &= 2 + 2\sin\ \theta \rightarrow \theta = 0 \\ r &= 2 + 2\sin(0) \rightarrow \sin\ 0 = 0 \\ r &= 2 + 2(0)\text{simplify} \\ r &= 2 + 0 \\ r &= 2 \end{aligned}$$

$$\begin{aligned} r &= 2 - 2\cos\ \theta \\ r &= 2 - 2\cos\ \theta \rightarrow \theta = 0 \\ r &= 2 + 2\cos(0) \rightarrow \cos\ 0 = 0 \\ r &= 2 - 2(1)\text{simplify} \\ r &= 2 - 2 \\ r &= 0 \end{aligned}$$

The coordinates represent the same point $(0,0)$.

$$r = 2 + 2\sin\ \theta$$
$$r = 2 - 2\cos\ \theta$$

$$\begin{aligned}
2 + 2\sin\ \theta &= 2 - 2\cos\ \theta \\
2 - 2 + 2\sin\ \theta &= 2 - 2\cos\ \theta \rightarrow \text{simplify} \\
2\sin\ \theta &= -2\cos\ \theta \rightarrow \div(2\cos\ \theta) \\
\frac{2\sin\ \theta}{2\cos\ \theta} &= \frac{-2\cos\ \theta}{2\cos\ \theta} \rightarrow \div\text{simplify} \\
\frac{\sin\ \theta}{\cos\ \theta} &= -1 \rightarrow \frac{\sin\ \theta}{\cos\ \theta} = \tan\ \theta \\
\tan\ \theta &= -1 \\
\tan^{-1}(\tan\ \theta) &= \tan^{-1}(1) \\
\theta &= \frac{\pi}{4}
\end{aligned}$$

The tangent function is negative in the 2nd and 4th quadrants.

2nd Quadrant

$$\begin{aligned}
&\theta = \pi - \frac{\pi}{4} \\
&\theta = \pi - \frac{\pi}{4} \rightarrow \text{common deno min ator} \\
&\theta = \frac{4\pi}{4} - \frac{\pi}{4} \rightarrow \text{simplify} \\
&\theta = \frac{3\pi}{4}
\end{aligned}$$

4th Quadrant

$$\begin{aligned}
&\theta = 2\pi - \frac{\pi}{4} \\
&\theta = 2\pi - \frac{\pi}{4}\text{common deno min ator} \\
&\theta = \frac{8\pi}{4} - \frac{\pi}{4} \rightarrow \text{simplify} \\
&\theta = \frac{7\pi}{4}
\end{aligned}$$

$$\begin{aligned}
&r = 2 + 2\sin\ \theta \\
&r = 2 + 2\sin\ \theta \rightarrow \theta = \frac{3\pi}{4} \\
&r = 2 + 2\sin\left(\frac{3\pi}{4}\right) \rightarrow \text{simplify} \\
&r = 2 + 2\sin\left(\frac{3\pi}{4}\right) \rightarrow \sin\left(\frac{3\pi}{4}\right) = 0.7071 \\
&r = 2 + 2(0.7071) \rightarrow \text{simplify} \\
&r \approx 3.41 \\
&r = 2 + 2\sin\ \theta \rightarrow \theta = \frac{7\pi}{4} \\
&r = 2 + 2\sin\ \theta\left(\frac{7\pi}{4}\right) \rightarrow \text{simplify} \\
&r = 2 + 2\sin\ \theta\left(\frac{7\pi}{4}\right) \rightarrow \sin\ \theta\left(\frac{7\pi}{4}\right) = -0.7071 \\
&r = 2 + 2(-0.7071) \rightarrow \text{simplify} \\
&r \approx 0.59
\end{aligned}$$

$$\begin{aligned}
&r = 2 - 2\cos\ \theta \\
&r = 2 - 2\cos\ \theta \rightarrow \theta = \frac{3\pi}{4} \\
&r = 2 - 2\cos\left(\frac{3\pi}{4}\right) \rightarrow \text{simplify} \\
&r = 2 - 2\cos\left(\frac{3\pi}{4}\right) \rightarrow \cos\left(\frac{3\pi}{4}\right) = (0.7071) \\
&r = 2 - 2(0.7071) \rightarrow \text{simplify} \\
&r \approx 3.41 \\
&r = 2 - 2\cos\ \theta \rightarrow \theta = \frac{7\pi}{4} \\
&r = 2 - 2\cos\ \theta\left(\frac{7\pi}{4}\right) \rightarrow \text{simplify} \\
&r = 2 - 2\cos\ \theta\left(\frac{7\pi}{4}\right) \rightarrow \cos\ \theta\left(\frac{7\pi}{4}\right) = -0.7071 \\
&r = 2 - 2(-0.7071) \rightarrow \text{simplify} \\
&r \approx 0.59
\end{aligned}$$

Substituting the points into the equation $r = 2 - 2\cos\ \theta$ is not necessary but it does confirm the points.

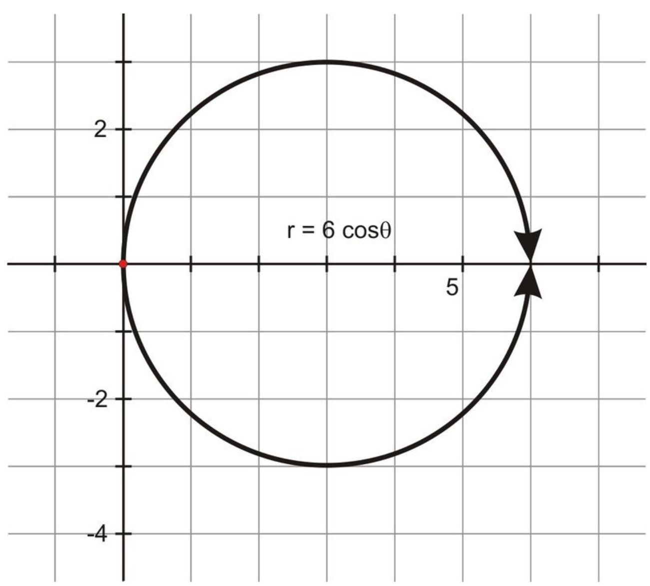
The **points of intersection** are $\left(3.41, \frac{3\pi}{4}\right), \left(0.59, \frac{7\pi}{4}\right)$ *and* $(0,0)$.

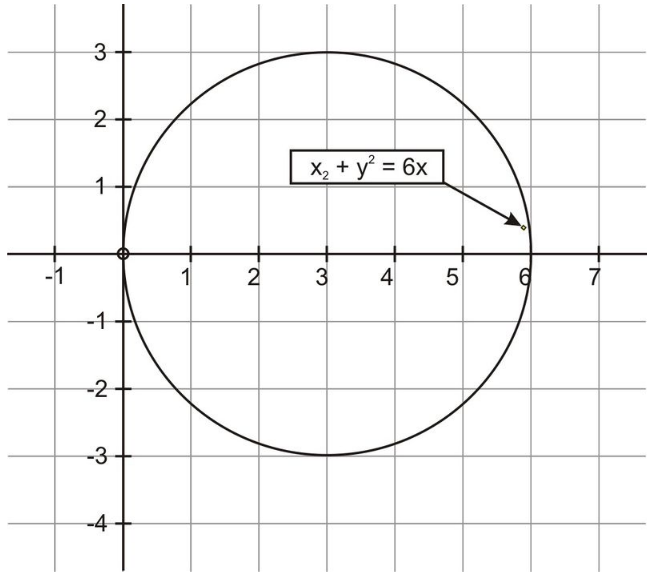
Equivalent Polar Curves

Review Exercises

1. To write the equation in polar form, use the formulas $r^2 = x^2 + y^2$ *and* $x = r\cos\ \theta$.

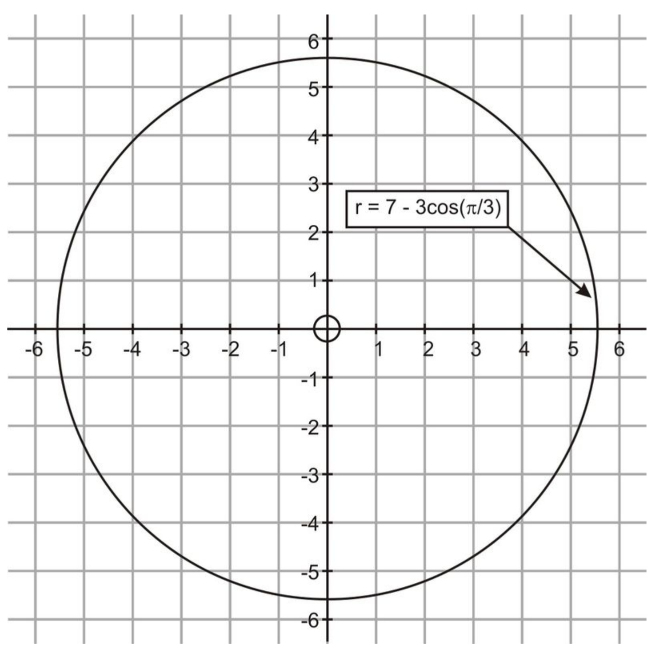
$$\begin{aligned}
x^2 + y^2 &= 6x \\
x^2 + y^2 &= 6x \rightarrow x^2 + y^2, x = r\cos\ \theta \\
r^2 &= 6(r\cos\ \theta) \rightarrow \text{simplify} \\
r^2 &= 6r\cos\ \theta \rightarrow \div(r) \\
\frac{\cancel{r^2}(r)}{\cancel{r}} &= \frac{6\cancel{r}\cos\ \theta}{\cancel{r}} \rightarrow \text{simplify} \\
r &= 6\cos\ \theta
\end{aligned}$$

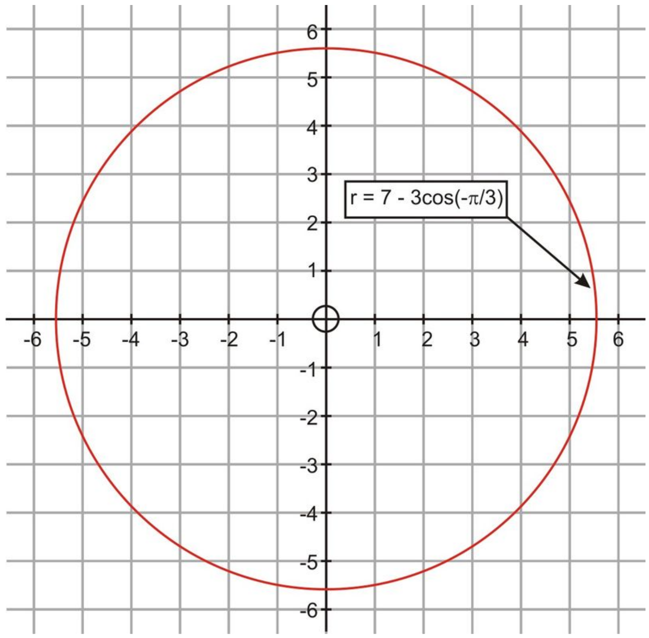




Both equations $r = \cos\ \theta$ and $x^2 + y^2 = 6x$ produced the same graph – a circle with center $(3,0)$ and a radius of 3 .

2. If the equations $r = 7 - 3\cos\left(\frac{\pi}{3}\right)$ and $r = 7 - 3\cos\left(-\frac{\pi}{3}\right)$ produce the same graph, then the equations are equivalent.





Yes, the both equations are equivalent. They are graphed above on separate axes but both could be plotted on the same grid. Only one graph would appear.

Recognize

Review Exercises

Recognize $i = \sqrt{-1}$, $\sqrt{-x} = i\sqrt{x}$

1. To express the square root of a negative number in terms of i , express the radicand as the product of a positive number and (-1) . Then write this product as the product of the square root of the positive factor and the square root of (-1) .

a)

$$\begin{aligned}
&\sqrt{-64}\\
&\sqrt{(64)(\sqrt{-1})}\\
&= 8i
\end{aligned}$$

b)

$$\begin{aligned}
&-\sqrt{-108}\\
&-\sqrt{(108)(-1)}\\
&(-\sqrt{108})(\sqrt{-1})\\
&(-\sqrt{(36)(3)})(\sqrt{-1})\\
&= -6i\sqrt{3}
\end{aligned}$$

c)

$$(\sqrt{-15})^2$$
$$((\sqrt{15})(\sqrt{-1}))^2$$
$$(i\sqrt{15})^2$$
$$= 15i^2 \rightarrow i^2 = -1$$
$$(-1)(15)$$
$$= -15$$

d)

$$(\sqrt{-49})(\sqrt{-25})$$
$$(\sqrt{(49)(-1)})(\sqrt{(25)(-1)})$$
$$(\sqrt{49})(\sqrt{-1})(\sqrt{25})(\sqrt{-1})$$
$$(7)(i)(5)(i)$$
$$= 35i^2 \rightarrow i^2 = -1$$
$$(-1)(35)$$
$$= -35$$

Standard Forms of Complex Numbers C

Review Exercises

1. To simplify each complex number means to write it in standard form $(a+bi)$. The conjugate is of the form $(a+bi)$ with the same [U+0080] [U+0098] a [U+0080] [U+0099] but the opposite [U+0080] [U+0098] bi [U+0080] [U+0099] . Example: The conjugate of $4-3i$ *is* $4+3i$.

a)

$$-\sqrt{1}-\sqrt{-400}$$
$$-\sqrt{1}-\sqrt{(400)(-1)}$$
$$-\sqrt{1}-(\sqrt{(400)})(\sqrt{-1}) \rightarrow \text{simplify}$$
$$= -1-20i$$
$$\text{The conjugate is } -1+20i$$

b)

$$\sqrt{-36i^2}+\sqrt{-36}$$
$$\sqrt{(36)(-1)}+\sqrt{(36)(-1)}$$
$$\sqrt{36}+(\sqrt{36})(\sqrt{-1}) \rightarrow \text{simplify}$$
$$6+6i^2$$
$$\text{The conjugate is } 6-6i$$

2. Solve the equation for the variables x and y .

$$\begin{aligned}
6i-7 &= x-yi \\
6i-7-6i+7 &= 3-x-yi-6i+7 \rightarrow \text{simplify} \\
0 &= 10-x-iy-6i \rightarrow \text{simplify} \\
0+x+yi &= 10-x-yi-6i+x+yi \rightarrow \text{simplify} \\
x+yi &= 10-6i \rightarrow \text{solve} \\
x &= 10 \text{and} yi = -6i \rightarrow \text{solve} \\
x &= 10 \\
yi &= -6i \rightarrow \div(i) \\
\frac{y\cancel{i}}{\cancel{i}} &= \frac{-6\cancel{i}}{\cancel{i}} \\
y &= -6
\end{aligned}$$

The values of $x = 10$ and $y = -6$ satisfy the equation $6i-7 = 3-x-yi$.

The Set of Complex Numbers (complex, real, irrational, rational, etc)

Review Exercises: None – Simply an information lesson

Complex Number Plane

Review Exercises

1. The absolute value of a complex number in standard form $(a+bi)$ is the square root of a^2+b^2 . In other words $|a+bi| = \sqrt{a^2+b^2}$.

a) The coordinates of the points plotted on the complex number plane are:

$$\begin{aligned}
&A(-5-3i) \\
&B(6+2i) && |6+2i| = \sqrt{40} \approx 6.3 \\
&C(2-5i) && |2-5i| = \sqrt{29} \approx 5.4 \\
&D(-2+4i) && |-2+4i| = \sqrt{20} \approx 4.5 \\
&E(3+6i)
\end{aligned}$$

The absolute values of the other 3 points are shown above. The detailed solutions for A and E are shown below. The question requires only two points to be done.

$$
\begin{aligned}
&A(-5-3i)\\
&A(-5-3i) \to a=-5, b=-3\\
|a+bi| &= \sqrt{a^2+b^2}\\
|-5-3i| &= \sqrt{a^2+b^2} \to (-5-3i) \to a=-5, b=-3\\
|-5-3i| &= \sqrt{(-5)^2+(-3)^2} \to \text{simplify}\\
|-5-3i| &= \sqrt{25+9} \to \text{simplify}\\
|-5-3i| &= \sqrt{34} \to \text{simplify}\\
|-5-3i| &= \sqrt{34} \approx 5.8\\
&E(3+6i)\\
&E(3+6i) \to\to a=3, b=6\\
|a+bi| &= \sqrt{a^2+b^2}\\
|3+6i| &= \sqrt{a^2+b^2} \to (3+6i) \to a=3, b=6\\
|3+6i| &= \sqrt{(3)^2+(6)^2} \to \text{simplify}\\
|3+6i| &= \sqrt{9+36} \to \text{simplify}\\
|3+6i| &= \sqrt{45} \to \text{simplify}\\
|3+6i| &= \sqrt{45} \approx 6.7
\end{aligned}
$$

Quadratic Formula

Review Exercises

1. To describe the nature of the roots, the value of the discriminant must be calculated. If the value of the discriminant b^2+4ac is less than zero, the roots will be a complex conjugate pair of roots.

Set the equation equal to zero.

$$
\begin{aligned}
&5x^2-x+5=6x+1\\
&5x^2-x+5-6x-1=6x+-6x-1 \to \text{simplify}\\
&5x^2-7x+4=0 \to \text{simplify}\\
&5x^2-7x+4=0 \to a=5, b=-7, c=4\\
&b^2-4ac\\
&b^2-4ac=(-7)^2-4(5)(4) \to \text{evaluate}\\
&b^2-4ac=-31\\
&b^2-4ac<0 \to \text{a complex conjugate pair of roots}
\end{aligned}
$$

Solve the equation using the quadratic formula: $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$

$$5x^2 - 7x + 4 = 0 \to a = 5, b = -7, c = 4$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-(-7) \pm \sqrt{(-7)^2 - 4(5)(4)}}{2(5)} \to \text{simplify}$$

$$x = \frac{7 \pm \sqrt{49 - 80}}{10} \to \text{simplify}$$

$$x = \frac{7 \pm \sqrt{-31}}{10} \to \text{simplify}$$

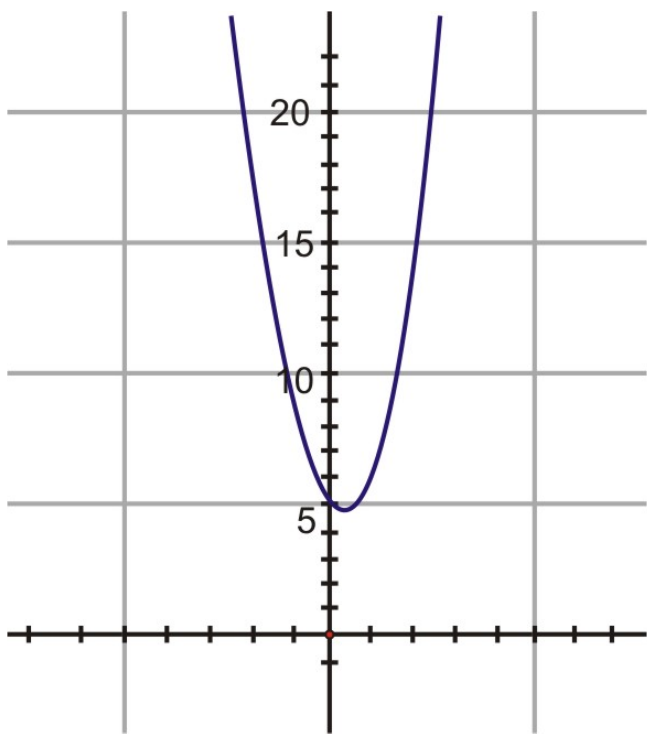
$$x = \frac{7 \pm (\sqrt{31})(\sqrt{-1})}{10} \to \text{solve}$$

$$x = \frac{7 + i\sqrt{31}}{10} \to \text{evalute} \qquad x = \frac{7 - i\sqrt{31}}{10} \to \text{evalute}$$

$$x = \frac{7 + 5.6i}{10} \to \text{evalute} \qquad x = \frac{7 - 5.6i}{10} \to \text{evalute}$$

$$x = \frac{7 + 5.6i}{10} \approx 0.7 + 0.56i \qquad x = \frac{7 - 5.6i}{10} \approx 0.7 - 0.56i$$

2.



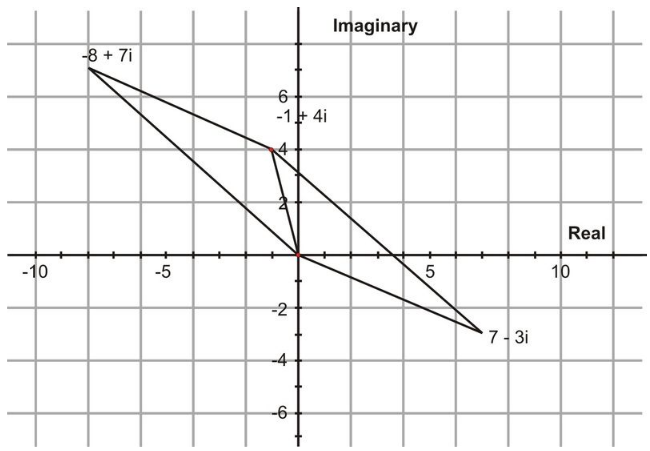
The above parabola does not intersect the $x-$ axis. This means that the value of the discriminant, $b^2 - 4ac$, will be less than zero. The roots of his quadratic function will be a complex, conjugate pair.

Sums and Differences of Complex Numbers

Review Exercises

1. The steps involved in adding and subtracting real numbers also apply to complex numbers. To subtract is actually adding the opposite and adding involves following the rules for integers.

a) Graphically



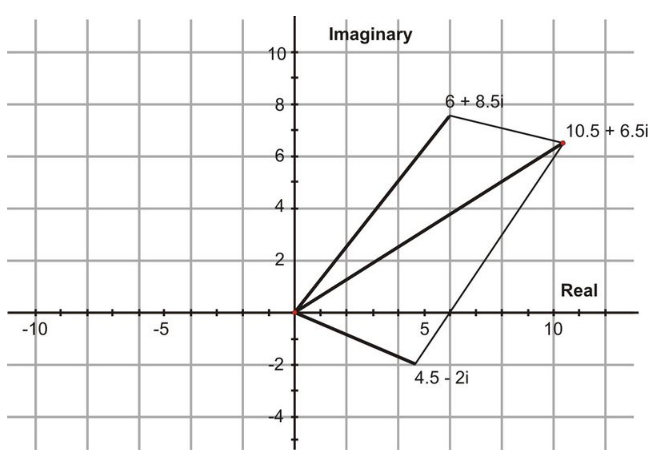
Subtract the complex numbers:

$$
\begin{aligned}
&(7-3i)-(8-7i)\\
&(7-3i)-(8-7i) \rightarrow \text{add}(-8+7i)\\
&(7-3i)-(-8+7i) \rightarrow \text{simplify}\\
&(7-8)+(-3i+7i) \rightarrow \text{simplify}\\
&=-1+4i
\end{aligned}
$$

Check:

$$
\begin{aligned}
&(7-3i)-(8-7i)\\
&(7-3i)-(-8+7i)\\
&(7-8)+(-3i+7)i\\
&=-1+4i
\end{aligned}
$$

b) Graphically:



Add the complex numbers:

$$(4.5-2.0i)+(6.0+8.5i)$$
$$(4.5+6.0)+(-2.0i+8.5i) \rightarrow \text{simplify}$$
$$(4.5+6.0)+(-2.0+8.5)i \rightarrow \text{simplify}$$
$$=10.5+6.5i$$

Check:

$$(4.5-2.0i)+(6.0+8.5i)$$
$$(4.5+6.0)+(-2.0+8.5)i$$
$$=10.5+6.5i$$

Products and Quotients of Complex Numbers (conjugates)

Review Exercises

1. To perform the operation of multiplication in part [U+0080][U+0098]*a*[U+0080][U+0099] , apply the distributive property and simplify the answer. In part [U+0080][U+0098]*b*[U+0080][U+0099] , multiply the numerator and the denominator of the fraction by the conjugate of the denominator. Apply the distributive property and simplify the answer.

a)

$$(7-5i)(4-9i)$$
$$(7-5i)(4-9i) \rightarrow \text{expand}$$
$$7(4-9i)-5i(4-9i) \rightarrow \text{distributive property}$$
$$28-63i-20i+45i^2 \rightarrow \text{simplify}$$
$$28-83i+45i^2 \rightarrow i^2=-1$$
$$28-83i-45(-1) \rightarrow \text{simplify}$$
$$28-83i-45 \rightarrow \text{simplify}$$
$$=-17-83i$$

b)

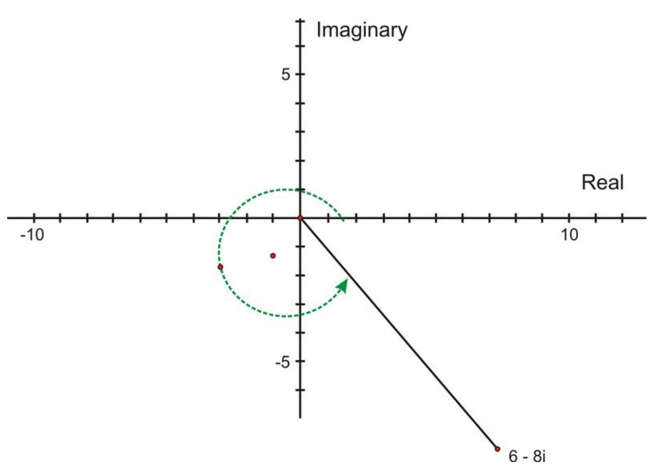
$$\frac{4+7i}{9-5i}$$
$$\frac{4+7i}{9-5i} \rightarrow \text{multiply by}(9+5i)$$
$$\left(\frac{4+7i}{9-5i}\right)\left(\frac{9+5i}{9+5i}\right) \rightarrow \text{simplify}$$
$$\frac{(4+7i)(9+5i)}{(9-5i)(9+5i)} \rightarrow \text{exp and}$$
$$\frac{4(9+5i)+7i(9+5i)}{9(9+5i)-5i(9+5i)}\text{distributive property}$$
$$\frac{36+20i+63i+35i^2}{81+45i-45i-25i^2} \rightarrow \text{simplify}$$
$$\frac{36+83i+35i^2}{81-25i^2} \rightarrow i^2=-1$$
$$\frac{36+83i+35(-1)}{81-25(-1)} \rightarrow \text{simplify}$$
$$\frac{36+83i-35}{81+25} \rightarrow \text{simplify}$$
$$\frac{1+83i}{106} \rightarrow \text{simplify}$$
$$\frac{1+83i}{106} \approx 0.009+0.783i$$

The Trigonometric or Polar Form of a Complex Number

$r\ cis\ \theta$

Review Exercises

1.To express the point $(6-8i)$ graphically, plot the point as you would the point $(6,-8)$. The $y-$ axis is the Imaginary axis and the $x-$ axis is the Real axis. To write $(6,-8)$ in its polar form, the value of [U+0080][U+0098]r[U+0080][U+0099] must be determined as well the measure of theta. In addition, $x = r\cos\ \theta$ and $y = r\sin\ \theta$.



$$\begin{aligned}
&6-8i \\
&x=6 \text{ and } y=-8 \\
&r=\sqrt{x^2+y^2} \rightarrow \text{det er min } e\text{the value of} r \\
&r=\sqrt{(6)^2+(-8)^2} \rightarrow \text{simplify} \\
&r=\sqrt{100} \rightarrow \text{simplify} \\
&r=10
\end{aligned}$$

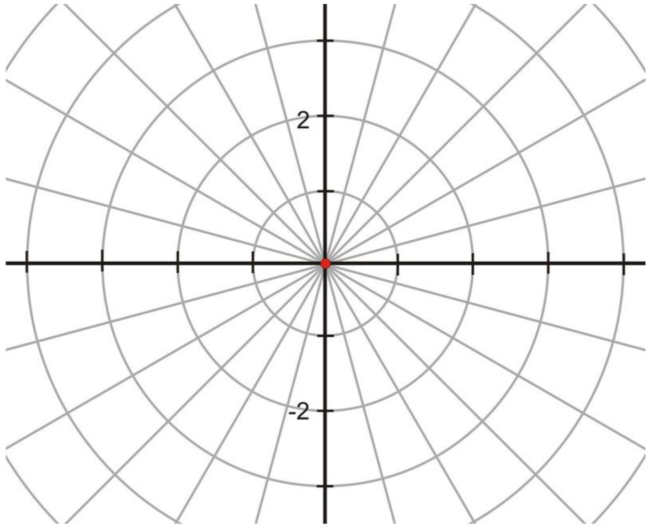
$$\begin{aligned}
\tan\ \theta &= \frac{\text{opp}}{\text{adj}} = \frac{y}{x} \\
\tan\ \theta &= \frac{-8}{6} \rightarrow \text{divide} \\
\tan\ \theta &= -1.3333 \\
\tan^{-1}(\tan\ \theta) &= \tan^{-1}(1.3333) \\
\theta &= 53.1^\circ
\end{aligned}$$

The tangent function is negative in the 4th quadrant and the point $6-8i$ is located there. The measure of θ is $360^\circ - 53.1^\circ = 306.9^\circ$

In polar form $6-8i$ is $10(\cos 306.9^\circ + i\ \sin 306.9^\circ)$ or $10\angle 306.9^\circ$.

2.

$$\begin{aligned}
&3\left(\cos\frac{\pi}{4}+i\ \sin\frac{\pi}{4}\right) \\
&r=3 \\
&x=\cos\frac{\pi}{4}=\frac{\sqrt{2}}{2} \\
&y=\sin\frac{\pi}{4}=\frac{\sqrt{2}}{2} \\
&3\left(\cos\frac{\pi}{4}+i\ \sin\frac{\pi}{4}\right) \text{ is actually} 3\left(\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}\right) \\
&=\frac{3\sqrt{2}}{2}+\frac{3\sqrt{2}}{2}i
\end{aligned}$$



De Moivre's Theorem

Review Exercises

1. The first step in solving this problem is to express the equation in polar form.

$$
\begin{aligned}
z &= -\frac{1}{2} + i\frac{\sqrt{3}}{2} \\
z &= -\frac{1}{2} + i\frac{\sqrt{3}}{2} x = -\frac{1}{2}, y = \left(\frac{\sqrt{3}}{2}\right) \\
r &= \sqrt{x^2 + y^2} \\
r &= \sqrt{\left(-\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)} \rightarrow \text{simplify} \\
r &= \sqrt{\frac{1}{4} + \frac{3}{4}} \rightarrow \text{simplify} \\
r &= \sqrt{1} = 1
\end{aligned}
$$

$$
\begin{aligned}
\tan\ \theta &= \frac{\text{opp}}{\text{adj}} = \frac{y}{x} \\
\tan\ \theta &= -\frac{\sqrt{3}}{1} \rightarrow \text{simplify} \\
\tan\ \theta &= -\sqrt{3} \\
\tan^{-1}(\tan\ \theta) &= \tan^{-1}(\sqrt{3}) \\
\theta &\approx 60^\circ
\end{aligned}
$$

The point is located in the 2^{nd} quadrant and the tangent function is negative here. The measure of theta is $180^\circ - 60^\circ = 120^\circ$.

The polar form of $z = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$ is $z = 1(\cos 120^\circ + i\ \sin 120^\circ)$.

Apply De Moivre's Theorem

$$\begin{aligned}
z^n &= [r(\cos\ \theta + i\ \sin\ \theta)]^n = r^n(\cos\ \theta + i\ \sin n\ \theta) \\
z^3 &= 1^3[\cos 3(120^\circ) + i\ \sin 3(120^\circ)] \\
z^3 &= 1^3(\cos 360^\circ + i\ \sin 360^\circ) \\
z^3 &= 1(1 + i(0)) \\
z^3 &= 1
\end{aligned}$$

2. To write the expression $[2(\cos 315^\circ + i\ \sin 315^\circ)]^3$ in rectangular form, simply work backwards and apply De Moivre's Theorem

$$[2(\cos 315^\circ + i\ \sin 315^\circ)]^3$$

$(\cos 315^\circ + i\ \sin 315^\circ)^3 \rightarrow r = 2$ and $\theta = 315^\circ \rightarrow \frac{7\pi}{4}$

$$\begin{aligned}
z^n &= [r(\cos\ \theta + i\ \sin\ \theta)]^n = r^n(\cos\ \theta + i\ \sin n\ \theta) \\
z^n &= 2^3\left(\cos 3\left(\frac{7\pi}{4}\right) + i\ \sin 3\left(\frac{7\pi}{4}\right)\right) \\
z^3 &= 8\left(\cos\frac{21\pi}{4} + i\ \sin\frac{21\pi}{4}\right) \rightarrow \frac{21\pi}{4}(\text{3rd quadrant}) \\
&\qquad \rightarrow \text{Both are negative} \\
z^3 &= 8\left(-\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right) \rightarrow \text{simplify} \\
z^3 &= \cancel{8}(4)\left(-\frac{\sqrt{2}}{\cancel{2}} - i\frac{\sqrt{2}}{\cancel{2}}\right) \rightarrow \text{simplify} \\
z^3 &= -4\sqrt{2} - 4i\sqrt{2}
\end{aligned}$$

nth Root Theorem

Review Exercises

1. To determine the cube root of $27i$, write it as a complex number, calculate the value of r and the measure of θ

$$\begin{aligned}
&\sqrt[3]{27i} \\
&\sqrt[3]{27i} \rightarrow (a + bi) \\
&(0 + 27i)^{\frac{1}{3}} \rightarrow a = 0 \text{ and } b = 27 \\
&\qquad\qquad \rightarrow x = 0 \text{ and } y = 27
\end{aligned}$$

Calculate the value of [U+0080][U+0098]r[U+0080][U+0099] :

$$\begin{aligned}
r &= \sqrt{x^2+y^2} \\
r &= \sqrt{(0)^2+(27)^2} \rightarrow \text{simplify} \\
r &= 27
\end{aligned}$$

$$\begin{aligned}
\theta &= \frac{\pi}{2} \\
\sqrt[3]{27i} &= \left[27\left(\cos\frac{\pi}{2}+i\ \sin\frac{\pi}{2}\right)\right]^{\frac{1}{3}} \rightarrow \text{simplify} \\
\sqrt[3]{27i} &= \left[\sqrt[3]{27}\left(\cos\left(\frac{1}{3}\right)\frac{\pi}{2}+i\ \sin\left(\frac{1}{3}\right)\frac{\pi}{2}\right)\right] \rightarrow \text{simplify} \\
\sqrt[3]{27i} &= 3\left(\cos\frac{\pi}{6}+i\ \sin\frac{\pi}{6}\right) \rightarrow \cos\frac{\pi}{6}=\frac{\sqrt{3}}{2}, \sin\frac{\pi}{6}=\frac{1}{2} \\
\sqrt[3]{27i} &= 3\left(\frac{\sqrt{3}}{2}+i\frac{1}{2}\right) \rightarrow \text{simplify} \\
\sqrt[3]{27i} &= 3\left(\frac{\sqrt{3}}{2}+\frac{1}{2}i\right)
\end{aligned}$$

2. To determine the principal root means to calculate the positive root. The square root of a number can be $\pm$ The principal root is the positive root only.

$$\begin{aligned}
&(1+i)^{\frac{1}{5}} \\
&(a+bi)^{\frac{1}{5}} \\
&(1+i)^{\frac{1}{5}} \rightarrow a=1,\ b=1 \\
&\qquad\quad \rightarrow x=,y=1
\end{aligned}$$

Calculate the value of [U+0080][U+0098]r[U+0080][U+0099]:

$$\begin{aligned}
r &= \sqrt{x^2+y^2} \\
r &= \sqrt{(1)^2+(1)^2} \rightarrow \text{simplify} \\
r &= \sqrt{2}
\end{aligned}$$

$$\tan\ \theta = \frac{\text{opp}}{\text{adj}} = \frac{y}{x}$$
$$\tan\ \theta = \frac{1}{1} \rightarrow \text{simplify}$$
$$\tan\ \theta = 1$$
$$\tan^{-1}(\tan\ \theta) = \tan^{-1}(1)$$
$$\theta \approx \frac{\sqrt{2}}{2} = \frac{\pi}{4} \qquad \text{polar From} : \left(\sqrt{2}, \frac{\pi}{4}\right)$$
$$(1+i)^{\frac{1}{5}} = \left[\sqrt{2}\left(\cos\frac{\pi}{4} + i\ \sin\frac{\pi}{4}\right)\right]^{\frac{1}{5}} \rightarrow \text{simplify}$$
$$(1+i)^{\frac{1}{5}} = \left[(\sqrt{2})^{\frac{1}{5}}\left(\cos\left(\frac{1}{5}\right)\frac{\pi}{4} + i\ \sin\left(\frac{1}{5}\right)\frac{\pi}{4}\right)\right] \rightarrow \text{simplify}$$
$$(1+i)^{\frac{1}{5}} = \sqrt[5]{2}\left(\cos\frac{\pi}{20} + i\ \sin\frac{\pi}{20}\right) \rightarrow \text{evaluate}$$

$(1+i)^{\frac{1}{5}} = (1.07+1.07i) \rightarrow s\tan\mathit{dard\,from}$. This is the principal root of $(1+i)^{\frac{1}{5}}$.

Solve Equations

Review Exercises:

1. To solve the equation $x^4+1=0$, an expression for determining the fourth roots of the equation, must be written. Calculate the value of [U+0080][U+0098]r[U+0080][U+0099] and the measure of θ .

$$x^4+1=0$$
$$x^4+1-1=0-1 \rightarrow \text{solve}$$
$$x^4=-1$$
$$x^4=-1+0i$$
$$x^4=-1+0i \rightarrow x=-1, y=0$$

Calculate the value of [U+0080][U+0098]r[U+0080][U+0099] :

$$r = \sqrt{x^2+y^2}$$
$$r = \sqrt{x^2+y^2} \rightarrow x=-1, y=0$$
$$r = \sqrt{(-1)^2+(0)^2} \rightarrow \text{simplify}$$
$$r = \sqrt{1} = 1$$

$$
\begin{aligned}
\tan\ \theta &= \frac{\text{opp}}{\text{adj}} = \frac{y}{x} \\
\tan\ \theta &= \frac{0}{-1} \to \text{simplify} \\
\tan\ \theta &= \left(\frac{0}{-1}\right) + \pi \\
\tan^{-1}(\tan\ \theta) &= \tan^{-1}\left(\frac{0}{-1}\right) + \pi \\
\theta &= \pi && \text{Polar Form: } (1, \pi)
\end{aligned}
$$

$$
\begin{aligned}
(-1+0i)^{\frac{1}{4}} &= [1(\cos(\pi+2\pi k)) + i\ \sin(\pi+2\pi k)]^{\frac{1}{4}} \\
(-1+0i)^{\frac{1}{4}} &= (1)^{\frac{1}{4}}\left(\cos\frac{\pi+2\pi k}{4} + i\ \sin\frac{\pi+2\pi k}{4}\right) \to k=0 \\
x_1 &= 1\left(\cos\frac{\pi}{4} + i\ \sin\frac{\pi}{4}\right) \to k=0 \\
(-1+0i)^{\frac{1}{4}} &= (1)^{\frac{1}{4}}\left(\cos\frac{\pi+2\pi k}{4} + i\ \sin\frac{\pi+2\pi k}{4}\right) \to k=1 \\
x_2 &= 1\left(\cos\frac{3\pi}{4} + i\ \sin\frac{3\pi}{4}\right) \to k=1 \\
(-1+0i)^{\frac{1}{4}} &= (1)^{\frac{1}{4}}\left(\cos\frac{\pi+2\pi k}{4} + i\ \sin\frac{\pi+2\pi k}{4}\right) \to k=2 \\
x_3 &= 1\left(\cos\frac{5\pi}{4} + i\ \sin\frac{5\pi}{4}\right) \to k=2 \\
(-1+0i)^{\frac{1}{4}} &= (1)^{\frac{1}{4}}\left(\cos\frac{\pi+2\pi k}{4} + i\ \sin\frac{\pi+2\pi k}{4}\right) \to k=3 \\
x_4 &= 1\left(\cos\frac{7\pi}{4} + i\ \sin\frac{7\pi}{4}\right) \to k=3
\end{aligned}
$$

$$
\begin{aligned}
x_1 &= 1\left(\cos\frac{\pi}{4} + i\ \sin\frac{\pi}{4}\right) \to \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2} \\
x_2 &= 1\left(\cos\frac{3\pi}{4} + i\ \sin\frac{3\pi}{4}\right) \to -\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2} \\
x_3 &= 1\left(\cos\frac{5\pi}{4} + i\ \sin\frac{5\pi}{4}\right) \to -\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2} \\
x_4 &= 1\left(\cos\frac{7\pi}{4} + i\ \sin\frac{7\pi}{4}\right) \to \frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}
\end{aligned}
$$